Lipschitz spaces over sets of positive measure

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Lipschitz spaces

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Fix a point $0 \in M$.

Given $f : M \to \mathbb{R}$, its Lipschitz constant is

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- If M and N are isometric or similar then $\mathrm{Lip}_0(M) \equiv \mathrm{Lip}_0(N)$
- If M and N are bi-Lipschitz equivalent then $\mathrm{Lip}_0(M) \sim \mathrm{Lip}_0(N)$

Lipschitz-free spaces

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Theorem

$\mathcal{F}(M)^* = \mathrm{Lip}_0(M)$, and the weak* topology on $B_{\mathrm{Lip}_0(M)}$ is the topology of pointwise convergence.

Consequently, an operator $T : \mathrm{Lip}_0(M) \to \mathrm{Lip}_0(N)$ has a preadjoint iff it is continuous wrt the pointwise topology.

Examples

- $\mathrm{Lip}_0(\mathbb{N}) \equiv \ell_\infty$:

$$\begin{aligned} T\colon\ \mathrm{Lip}_0(\mathbb{N}) &\to \ell_\infty \\ f &\mapsto \big(f(n) - f(n-1)\big)_{n=1}^{\infty} \end{aligned}$$

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Examples

Theorem *(Godard 2010)*

Let $M \subset \mathbb{R}$ closed and infinite.

- If M has null measure then $\mathcal{F}(M) \equiv \ell_1$.
- If M has positive measure then $\mathcal{F}(M) \sim L_1$.
  In fact, $\mathcal{F}(M) \equiv L_1 \oplus_1 \ell_1^n$ for some $0 \leq n \leq \infty$.

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What about higher dimension?

Theorem *(Naor, Schechtman 2007)*

$\mathrm{Lip}_0(\mathbb{R}^n)$ is not isomorphic to ℓ_∞ for $n \geq 2$.

It is unknown whether $\mathrm{Lip}_0(\mathbb{R}^n) \sim \mathrm{Lip}_0(\mathbb{R}^m)$ for any $n > m \geq 2$.

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$\mathrm{Lip}_0(\mathbb{Z}^n) \sim \mathrm{Lip}_0(\mathbb{R}^n)$.

$\mathrm{Lip}_0(M) \sim \mathrm{Lip}_0(\mathbb{R}^n)$ if $M \subset \mathbb{R}^n$ is "n-dimensional enough".

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Motivating question

What about $M \subset \mathbb{R}^n$ with positive (n-dimensional) measure?

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What about $M \subset \mathbb{R}^n$ with positive (n-dimensional) measure?

Main theorem *(Aliaga 2026)*

If $M \subset \mathbb{R}^n$ has positive Lebesgue measure then $\mathrm{Lip}_0(M) \sim \mathrm{Lip}_0(\mathbb{R}^n)$.

Pełczyński's decomposition method

Theorem *(Pełczyński, I presume?)*

Let X and Y be Banach spaces. Suppose that

- $X \overset{c}{\hookrightarrow} Y$
- $Y \overset{c}{\hookrightarrow} X$
- $X \sim \left(\bigoplus_{\mathbb{N}} X\right)_{\ell_p}$ for some $1 \leq p \leq \infty$

Then $X \sim Y$.

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Typically used with $p = 1$ for $\mathcal{F}(M)$ and $p = \infty$ for $\mathrm{Lip}_0(M)$.

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Open question

Is $\mathcal{F}(M) \sim (\bigoplus_{\mathbb{N}} \mathcal{F}(M))_{\ell_1}$ for every infinite M?

Auxiliary results

Separation lemma

If $M_n \subset M$ are well separated then $\mathcal{F}(\bigcup_n M_n) \sim (\bigoplus_n \mathcal{F}(M_n))_{\ell_1}$.

M_n are **well separated** if they are disjoint and there are $x_0 \in M$ and $C < \infty$ such that

$$d(x,x_0) + d(y,x_0) \leq C \cdot d(x,y)$$

for x,y from different sets M_n. Equivalently,

$$d'(x,y) = \begin{cases} d(x,y) & , x,y \text{ from the same set} \\ d(x,x_0) + d(y,x_0) & , x,y \text{ from different sets} \end{cases}$$

is a bi-Lipschitz equivalent metric on $\bigcup_n M_n$.

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Extension lemma

If $N \subset M \subset \mathbb{R}^n$ then $\mathcal{F}(N) \overset{c}{\hookrightarrow} \mathcal{F}(M)$.

That's because there is a bounded linear extension operator

$$E : \mathrm{Lip}_0(N) \to \mathrm{Lip}_0(M)$$

that is moreover continuous for the pointwise topology.
This doesn't hold outside $\mathbb{R}^n$ in general!

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Density lemma

If $M_n \subset M$ and for every $x \in M$ there are $x_n \in M_n$ with $x_n \to x$, then $\mathrm{Lip}_0(M) \overset{c}{\hookrightarrow} \left(\bigoplus_n \mathrm{Lip}_0(M_n)\right)_{\ell_\infty}$.

This one only works for Lipschitz spaces!

Proof of the main result

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Choose (p_n) points of density of M. Pick balls $B_n = B(p_n, r_n)$ so that

- M is very dense in B_n
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By Pełczyński, $\mathrm{Lip}_0(M) \sim \mathrm{Lip}_0(\mathbb{R}^n)$.

More generally...

Theorem *(Aliaga 2026)*

If $M \subset \mathbb{R}^n$ is not porous then $\mathrm{Lip}_0(M) \sim \mathrm{Lip}_0(\mathbb{R}^n)$.

Definition:

$M \subset \mathbb{R}^n$ is **porous** if there exists $\lambda \in (0,1)$ such that every ball in $\mathbb{R}^n$ with radius r contains a “hole” of radius λr that doesn't intersect M.

- porous sets are meager and Lebesgue null
- nets like $\mathbb{Z}^n$ are non-porous

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Lemma *(Aliaga 2026)*

If $M \subset \mathbb{R}^n$ is not porous then there exists a sequence $B(p_n, r_n)$ of balls in $\mathbb{R}^n$ such that

- $B_n \cap M$ is $\varepsilon_n r_n$-dense in B_n for some $\varepsilon_n \to 0$
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Question time

Open question

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Classify the spaces $\mathrm{Lip}_0(M)$ for $M \subset \mathbb{R}^2$ infinite. In particular:

- Is $\mathrm{Lip}_0(M)$ isomorphic to either ℓ_∞ or $\mathrm{Lip}_0(\mathbb{R}^2)$?
- Is $\mathrm{Lip}_0(M) \sim \mathrm{Lip}_0(\mathbb{R}^2)$ iff M is not porous?

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Theorem

A set $M \subset \mathbb{R}^n$ is not porous iff its Nagata dimension is n.

Thank you for your attention to this matter!

References:

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