Recent progress in Banach lattices

Antonio Avilés
Universidad de Murcia
with C. Rosendal, M. Taylor and P. Tradacete
(assisted by ChatGPT)

Proyecto PID2021-122126NB-C32 financiado por MICIU/AEI /10.13039/501100011033/ y
por FEDER Una manera de hacer Europa
y 21955/PI/22 by Fundación Séneca, ACyT Región de Murcia

Castro Urdiales 2026

Banach lattices

Definition

A Banach lattice is a Banach space with a partial order $\leq$ that makes it a lattice, with compatibility axioms (all identities in $\mathbb{R}$).

Free Banach lattices

1. For set A,
   $FBL(A)$ is the completion of all formal vector lattice expressions from A with maximal lattice norm for $A \subseteq B_X$
2. For Banach space E,
   $FBL[E]$ is made similarly.

Projective Banach lattices

Definition

A Banach lattice P is λ-projective if every $T : P \longrightarrow X/I$ lifts to $\tilde{T} : P \longrightarrow X$ with $\|\tilde{T}\| \leq \lambda \|T\|$.

$FBL(A)$, ℓ_1, Fin-dimensional, $C(K)$ for K ANR (polyhedra, $[0,1]^A$,...)

Projective Banach lattices

Definition

A Banach lattice P is λ-projective if every $T : P \longrightarrow X/I$ lifts to $\tilde{T} : P \longrightarrow X$ with $\|\tilde{T}\| \leq \lambda \|T\|$.

$FBL(A)$, ℓ_1, Fin-dimensional, $C(K)$ for K ANR (polyhedra, $[0,1]^A$,...)

A λ-projective that is not 1^+-projective:

- Renorm $C[0,1]$ with $\|f\|' = \|f\| + \alpha |f(0)|$.

The key: Azouzi-Ben Rjeb-Tradacete that 1^+-projective have the strong Nakano property. Norm is continuous for up-directed suprema.

Bossard theory

Question

Bossard theory allows to view separable Banach spaces as points in a Borel space, and compute complexity of classes. What about Banach lattices?

Bossard theory

Question

Bossard theory allows to view separable Banach spaces as points in a Borel space, and compute complexity of classes. What about Banach lattices?

Three ways to code separable Banach spaces:

Bossard theory

Question

Bossard theory allows to view separable Banach spaces as points in a Borel space, and compute complexity of classes. What about Banach lattices?

Three ways to code separable Banach spaces:

1. As closed subspaces of $C[0,1]$, endowed with the Effros-Borel structure.

Bossard theory

Question

Bossard theory allows to view separable Banach spaces as points in a Borel space, and compute complexity of classes. What about Banach lattices?

Three ways to code separable Banach spaces:

1. As closed subspaces of $C[0,1]$, endowed with the Effros-Borel structure.
2. As quotients of ℓ_1, and endow the Effros-Borel structure of subspaces of ℓ_1.

Question

Bossard theory allows to view separable Banach spaces as points in a Borel space, and compute complexity of classes. What about Banach lattices?

Three ways to code separable Banach spaces:

1. As closed subspaces of $C[0,1]$, endowed with the Effros-Borel structure.
2. As quotients of ℓ_1, and endow the Effros-Borel structure of subspaces of ℓ_1.
3. As completions of seminorms in $c_{00}(\mathbb{Q})$, that form a Polish space inside $\mathbb{R}^{c_{00}(\mathbb{Q})}$

The Bossard approach

Theorem (Leung, Lei, Oikhberg, Tursi 2019)

$C(2^{\mathbb{N}}, L^1)$ contains all separable Banach lattices.

Gramcko-Tursi thesis: We can code separable Banach lattices with the Effros-Borel structure of $C(2^{\mathbb{N}}, L^1)$ and study complexity of classes and equivalence relations (e.g. isomorphism classes and isomorphism relations).

Theorem (De Pagter, Wickstead)

All separable Banach lattices are quotients of the free Banach lattice on $\aleph_0$ generators $FBL(\mathbb{N})$.

Proposition

All separable Banach lattices come from completing lattice seminorms on the free $\mathbb{Q}$-vector lattice $FVL_{\mathbb{Q}}(\mathbb{N})$.

The three codings

Theorem

The three codings of separable Banach lattices are compatible,

1. *as sublattices of the embedding-universal* $C(2^{\mathbb{N}}, L^1)$,
2. *as quotients of the quotient-universal* $FBL(\mathbb{N})$,
3. *as lattice seminorms on* $FVL_{\mathbb{Q}}(\mathbb{N})$.

The last coding (Cúth-Doucha-Doležal-Kurka style) is finer since it is topological.

Complexity of projective

Theorem

The class of λ-projective Banach lattices is complete analytic.

Theorem

The class of ∞-projective Banach lattices is complete analytic.

Complexity of projective

Theorem

The class of λ-projective Banach lattices is complete analytic.

Theorem

The class of ∞-projective Banach lattices is complete analytic.

Analyticity comes naturally from definition. Completeness from a construction of Kurka based on Argyros-Deliyanni.

Complexity of projective

Theorem

The class of λ-projective Banach lattices is complete analytic.

Theorem

The class of ∞-projective Banach lattices is complete analytic.

Analyticity comes naturally from definition. Completeness from a construction of Kurka based on Argyros-Deliyanni.

The class of 1^+-projective is analytic. Is it complete analytic? Is it Borel?

Complexity of projective

Theorem

The class of λ-projective Banach lattices is complete analytic.

Theorem

The class of ∞-projective Banach lattices is complete analytic.

Analyticity comes naturally from definition. Completeness from a construction of Kurka based on Argyros-Deliyanni.

The class of 1^+-projective is analytic. Is it complete analytic? Is it Borel?

The class of ANR compacta is Borel (Dobrowolski, Rubin), hence projective $C(K)$ are Borel.

A problem

Problem (A., Martínez Cervantes and Rodríguez Abellán)
For which E is $FBL[E]$ projective?

$$E \cong \ell_1(A) \;\Rightarrow\; FBL[E] \text{ projective } \;\Rightarrow\; E \text{ is Schur}$$

A problem

Problem (A., Martínez Cervantes and Rodríguez Abellán)
For which E is $FBL[E]$ projective?

$$E \cong \ell_1(A) \;\Rightarrow\; FBL[E] \text{ projective} \;\Rightarrow\; E \text{ is Schur}$$

Theorem

If E is an ℓ_1-sum of finite dimensional Banach lattices, then $FBL[E]$ is 1^+-projective.

A problem

Problem (A., Martínez Cervantes and Rodríguez Abellán)
For which E is $FBL[E]$ projective?

$$E \cong \ell_1(A) \ \Rightarrow\ FBL[E] \text{ projective } \ \Rightarrow\ E \text{ is Schur}$$

Theorem

If E is an ℓ_1-sum of finite dimensional Banach lattices, then $FBL[E]$ is 1^+-projective.

Theorem

$FBL : SepBan \longrightarrow SepBanLat$ can be seen as a Borel function.

A problem

Problem (A., Martínez Cervantes and Rodríguez Abellán)
For which E is $FBL[E]$ projective?

$$E \cong \ell_1(A) \;\Rightarrow\; FBL[E] \text{ projective } \;\Rightarrow\; E \text{ is Schur}$$

Theorem

If E is an ℓ_1-sum of finite dimensional Banach lattices, then $FBL[E]$ is 1^+-projective.

Theorem

$FBL : SepBan \longrightarrow SepBanLat$ can be seen as a Borel function.

Projective is analytic, but By Braga (+Kurka) Schur is not analytic.
Therefore $FBL^{-1}(projective) \neq Schur$.

None of the implications can be reversed.