Combinatorics in nonseparable Banach spaces

Christina Brech

Instituto de Matemática e Estatística
Universidade de São Paulo

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Notation

X is an infinite dimensional Banach space

B_X is the closed unit ball of X

S_X is the unit sphere of X

X^* is the topological dual of X

K is a compact Hausdorff space

$C(K)$ is the Banach space of continuous scalar-valued functions on K

Recall that $w(K) = d(C(K))$.

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Partial positive answers:

- Banach spaces containing c_0 (Bessaga, Pełczyński, '58);
- Reflexive spaces (Pełczyński, '64);
- Weakly compactly generated spaces (Amir, Lindenstrauss, '68);
- Spaces whose dual contains ℓ_1 (Hagler, Johnson, '77).

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Theorem (Johnson, Rosenthal, '72)

Every separable space X contains a closed infinite dimensional linear subspace Y of X such that X/Y is infinite dimensional.

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- For $f, g : \mathbb{N} \to \mathbb{N}$, write $f \leq^* g$ if

$$\{n \in \mathbb{N} : f(n) > g(n)\} \text{ is finite.}$$

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- The bounding number $\mathfrak{b}$ is the least cardinality of an unbounded family $\mathcal{F} \subseteq \mathbb{N}^{\mathbb{N}}$; equivalently, there is no $g : \mathbb{N} \to \mathbb{N}$ such that $f \leq^* g$ for every $f \in \mathcal{F}$.

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Question (Davis, Johnson, '73)

Does every Banach space have a bounded fundamental biorthogonal system?

Biorthogonal systems

Definition

A family $(x_\alpha, x^*_\alpha)_{\alpha\in\kappa} \subseteq X \times X^*$ is a biorthogonal system if $x^*_\alpha(x_\beta) = 0$ if $\beta \neq \alpha$ and $x^*_\alpha(x_\alpha) = 1$.
The system is:

- bounded if $\|x^*_\alpha\| \cdot \|x_\alpha\| \leq M$ for some M;
- fundamental if $\overline{span}\{x_\alpha : \alpha < \kappa\} = X$.

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Theorem (Plichko, '80)

If Γ is an index set of cardinality greater than $\mathfrak{c}$, then $\ell_\infty^c(\Gamma)$ does not admit a bounded fundamental biorthogonal system.

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Remark

We can always construct a bounded system with triangular matrix.

$C(K)$ spaces

- If $(p_\alpha)_{\alpha\in\kappa} \subseteq K$ is discrete, then there are $(f_\alpha)_{\alpha\in\kappa} \subseteq C(K)$ such that $(f_\alpha, \delta_{p_\alpha})_{\alpha\in\kappa}$ is a biorthogonal system.

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Proposition (Zenor, '80; Velichko, '81)

If K is scattered and K^n is hereditarily separable for every $n \in \mathbb{N}$, then $C(K)$ is weakly hereditarily Lindelöf. Hence, $C(K)$ has no uncountable biorthogonal systems.

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Proposition (Todorcevic, '06)

If K has a nonseparable subspace, then $C(K)$ contains an uncountable biorthogonal system.

Claim

There are distinct $p_\alpha, q_\alpha \in S \setminus \overline{\{p_\beta, q_\beta : \beta < \alpha\}}$ such that $f_\beta(p_\alpha) = f_\beta(q_\alpha)$ for every $\beta < \alpha$.

Further ZFC results

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If $C(K)$ has density greater than ω_1, then it has an uncountable semi-biorthogonal system.

Theorem (Koszmider, '09; Lopez-Abad, Todorcevic, '11)

There is a nonmetrizable compact space K such that $C(K)$ has no uncountable semi-biorthogonal systems.

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Negative results

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Lemma (Δ-system lemma)

Given an uncountable family $(F_\alpha)_{\alpha<\omega_1}$ of finite sets, there is an uncountable subfamily $(F_{\alpha_\xi})_{\xi<\omega_1}$ such that $|F_{\alpha_\xi}| = |F_{\alpha_\eta}|$ and $F_{\alpha_\xi} \cap F_{\alpha_\eta} = \Delta$ for every $\xi < \eta < \omega_1$.

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Theorem (Namioka, Phelps, '75)

K is scattered iff $C(K)$ is Asplund.