Combinatorics in nonseparable Banach spaces

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Summary of previous talks

- If $d(X) < \mathfrak{b}$, then X has a separable quotient.
- If $\mathfrak{b} = \omega_1$, then there is a nonmetrizable scattered K such that $C(K)$ has no uncountable biorthogonal systems.
- Under PFA, every nonseparable Banach space has an uncountable biorthogonal system.

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- If $\mathfrak{b} = \omega_1$, then there is a nonmetrizable scattered K such that $C(K)$ has no uncountable biorthogonal systems.
- Under PFA, every nonseparable Banach space has an uncountable biorthogonal system.

Theorem (B., Todorcevic)

Under the P-ideal dichotomy, if $\mathfrak{b} > \omega_1$, then every nonseparable Asplund space has an uncountable biorthogonal system.

Proof overview

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- Apply Hájek-Johanis' result.

Positive result

Theorem (Hájek, Johanis, '20)

If there is a family $(z_\alpha, x^*_\alpha)_{\alpha\in\omega_1} \subseteq X \times X^*$ *such that* $x^*_\alpha(z_\alpha) = 1$ *and*

$$\forall \beta \in \omega_1 \quad (x^*_\alpha(z_\beta) : \alpha < \omega_1) \in \ell_1(\omega_1),$$

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- For every $k \in \mathbb{N}$, $\|y^\alpha_{k+1} - y^\alpha_k\| \leq C \cdot \delta^k$.
- $x_\alpha = \lim_k y^\alpha_k$.

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P-ideals

Definition

A P-ideal $\mathcal{I}$ of countable infinite subsets of some uncountable set S is an ideal of sets (ie. closed under subsets and finite unions) such that if $(I_n)_n$ is a sequence of elements of $\mathcal{I}$, then there is $I \in \mathcal{I}$ such that $I_n \setminus I$ is finite for every $n \in \mathbb{N}$.

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Example

If a family $\mathcal{F}$ of subsets of a set S is such that $|\mathcal{F}| < \mathfrak{b}$, then

$$\mathcal{F}^{\perp} := \{I \in [S]^{\omega} : (\forall F \in \mathcal{F}) \quad |F \cap I| < \omega\}$$

is a P-ideal.

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Remark

- PID is a consequence of the proper forcing axiom (PFA).
- PID is relatively consistent with the generalized continuum hypotesis (GCH).

The use of PID

- If $(f_\alpha(x) : \alpha < \omega_1) \in c_0(\omega_1)$ for every $x \in D$, then

$$\mathcal{I}_2 = \{I \in [\omega_1]^\omega : \forall x \in D \sum_{\alpha \in I} |f_\alpha(x)| < \infty\}$$

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- In this case, the second alternative cannot hold: every sequence converging weakly* to 0 has an absolutely summable subsequence.

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- We cannot guarantee that the second alternative cannot hold.

Use of the PID

- If the second alternative holds, assume that

$$\forall I \in \mathcal{I}_1 \quad \omega_1 \cap I \text{ is finite.}$$

Hence, $0 \in \overline{\{h_\alpha : \alpha < \omega_1\}}^{w^*}$.

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Lemma

In a compact Hausdorff space of weight strictly smaller than $\mathfrak{b}$, the sequential closure of any set is formed by the limit of all of its sequences.

Use of the PID

- Let $S = \{g - f : f \neq g \text{ and } f, g \in K\}$ and

  $\mathcal{I}_3 = \{I \in [S]^\omega : \forall x \in D \ \forall \varepsilon > 0 \ |\{h \in I : |h(x)| \geq \varepsilon\}| < \infty\}$

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We prove the following:

Claim

If $S = \bigcup_{n \in \mathbb{N}} S_n$, then 0 is in the w^*-sequential closure of some S_{n_0}.

Comments on the proof

The following ingredients are used:

- $\mathcal{F}$ is a maximal filter of w*-sequentially closed subsets of K containing the sequential closure of every weak*-open neighborhood of 0.

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  - If $Z \in \mathcal{F}^+$, then its w*-sequential closure is in $\mathcal{F}$.
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  - If $S = \bigcup_{n\in\mathbb{N}} S_n$, then a Fubini-type argument shows that there are $Z \in \mathcal{F}^+$ and n_0 such that

$$\forall f \in Z \quad \{g \in K : g - f \in S_{n_0}\} \in \mathcal{F}^+.$$

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Theorem (Lopez-Abad, Todorcevic, '11)

Consistently, there is a nonseparable Banach space which has no uncountable biorthogonal system, but has an uncountable ε-biorthogonal system for every $\varepsilon \in (0, 1)$.

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Under PID and $\mathfrak{b} > \omega_1$, does the dual of every Asplund space of density ω_1 have a fundamental biorthogonal system?

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If the dual of a Banach space X has a fundamental biorthogonal system such that the functionals belong to X, then X has a MIP renorming.

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Theorem (Jiménez-Sevilla, Moreno, '97)

If a nonseparable Banach space has the MIP, then there is a bounded family $(x_\alpha)_{\alpha<\omega_1}$ in X such that $x_\alpha \notin \overline{\mathit{conv}}\{x_\beta : \beta \neq \alpha\}$.

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See some of you in Brazil soon!

Obrigada!