On composition operators in some sequence spaces

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Space $bv_p(E)$

By $bv_p(E)$ for $p \geq 1$ (E a normed space), we denote the space of all sequences $x\colon \mathbb{N} \to E$ such that $\sum_{n=1}^{\infty} \|x(n+1) - x(n)\|^p < +\infty$.

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$$bv_1(E) \subset l^\infty(E), \quad bv_p(E) \not\subseteq l^\infty(E) \quad l^\infty(E) \not\subseteq bv_p(E)$$

Composition operator

Let E be a normed space and let Ω be a non-empty set. For given map $f : E \to E$ we define the *composition operator* C_f, generated by f, as

$$C_f(x)(t) = f(x(t)), \qquad \text{for } t \in \Omega,$$

where $x : \Omega \to E$.

Acting conditions with target space $bv_p(E)$

Theorem

Let $p, q \in [1, +\infty)$.

1. *If C_f maps $X \in \{bv_p(E), c(E), l^\infty(E)\}$ into $bv_q(E)$, then f is continuous on E.*

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1. *If C_f maps $X \in \{bv_p(E), c(E), l^\infty(E)\}$ into $bv_q(E)$, then f is continuous on E.*
2. *If C_f maps $Y \in \{l^p(E), c_0(E)\}$ into $bv_q(E)$, then f is continuous at 0.*

Acting conditions with target space $bv_p(E)$

Theorem

Let $p \in [1, +\infty)$. The composition operator C_f maps $c_0(E)$ into $bv_p(E)$ if and only if its generator f is locally constant at 0, that is, there exists a number $\delta > 0$ such that $f|_{\bar{B}_E(0,\delta)}$ is constant.

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Theorem

Let $p \in [1, +\infty)$ and let $X \in \{c(E), l^\infty(E)\}$. The composition operator C_f maps X into $bv_p(E)$ if and only if f is a constant function.

Acting conditions with target space $bv_p(E)$

Theorem

For any $p, q \in [1, +\infty)$ the following conditions are equivalent:

1. *the composition operator C_f maps $l^p(E)$ into $bv_q(E)$,*

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For any $p, q \in [1, +\infty)$ the following conditions are equivalent:

1. *the composition operator C_f maps $l^p(E)$ into $bv_q(E)$,*
2. *there exist $\delta > 0$ and $L \geq 0$ such that* $\|f(u) - f(w)\| \leq L\|u\|^{\frac{p}{q}} + L\|w\|^{\frac{p}{q}}$ *for all $u, w \in \bar{B}_E(0, \delta)$,*

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3. *there exist $\delta > 0$ and $L \geq 0$ such that $\|f(u) - f(0)\| \leq L\|u\|^{\frac{p}{q}}$ for every $u \in \bar{B}_E(0, \delta)$.*

Local boundedness

Theorem

Let $p \in [1, +\infty)$ and assume that the composition operator C_f maps $c_0(E)$ into $bv_p(E)$. Then, the following conditions are equivalent:

1. *C_f is bounded,*
2. *C_f is locally bounded,*
3. *f is a constant map.*

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Let $p, q \in [1, +\infty)$ and assume that the composition operator C_f maps $l^p(E)$ into $bv_q(E)$. Then, C_f is locally bounded if and only if f is.

Continuity

Let $p, q \in [1, +\infty)$, $X \in \{l^p(E), c_0(E), c(E), l^\infty(E), bv_p(E)\}$, $Y \in \{l^q(E), c_0(E), c(E), l^\infty(E), bv_q(E)\}$.

Proposition

Assume that the composition operator C_f maps X into Y. If C_f is continuous (respectively, locally uniformly continuous or uniformly continuous), then the function f possesses the same continuity property.

Pointwise continuity (with target space $bv_p(E)$)

Let $p, q \in [1, +\infty)$.

Theorem

Assume that the composition operator C_f maps:

-$c_0(E)$ into $bv_p(E)$ or

- $l^q(E)$ into $bv_p(E)$.

Then C_f is continuous if and only if f is.

Acting conditions between bv_p-spaces

Theorem

Let $p, q \in [1, +\infty)$. If the composition operator C_f maps $bv_p(E)$ into $bv_q(E)$, then f is Hölder continuous on compact subsets of E with exponent $\frac{p}{q}$.

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Corollary

Let $1 \leq q < p < +\infty$. The composition operator C_f acts between $bv_p(E)$ and $bv_q(E)$ if and only if f is a constant map.

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Theorem

Let $p \in [1, +\infty)$ and let E be a Banach space. The composition operator C_f maps $bv_1(E)$ into $bv_p(E)$ if and only if f is Hölder continuous on compact subsets of E with exponent $\frac{1}{p}$.

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Example

Let $a := (1, \frac{1}{2^2}, \frac{1}{3^2}, \dots)$ and for $u \in c_{00}(\mathbb{R})$ let $f(u) = \frac{u}{\|u-a\|_\infty}$. f is Lipschitz continuous on compact subsets of $c_{00}(\mathbb{R})$. However, it does not generate a composition operator acting between $bv_1(c_{00}(\mathbb{R}))$ and $bv_1(c_{00}(\mathbb{R}))$. Define $x \colon \mathbb{N} \to c_{00}(\mathbb{R})$ by $x(n) := P_n(a)$. Since $\|x(n+1) - x(n)\|_\infty = (n+1)^{-2}$ for $n \in \mathbb{N}$, we see that $x \in bv_1(c_{00}(\mathbb{R}))$. But,

$$\sum_{n=1}^{\infty} \|f(x(n+1)) - f(x(n))\|_\infty = +\infty.$$

Hence, $C_f(x) \notin bv_1(c_{00}(\mathbb{R}))$.

Local boundedness (between bv_p-spaces)

Remark

Local boundedness of $C_f : bv_1(E) \to bv_p(E)$ (E Banach space) is equivalent to local Hölder continuity of f with exponent $1/p$.

Example

Let $f : l^2(\mathbb{R}) \to l^2(\mathbb{R})$ be such that

$$f(u) = (\sin \psi(u), 0, 0, \ldots)$$

, where

$$\psi(u) = \sup\{(2\pi + 1)n|u(n)| - n : n \in \mathbb{N}\}.$$

f is Lipschitz continuous on compact subsets of $l^2(\mathbb{R})$, so $C_f : bv_1(l^2(\mathbb{R})) \to bv_1(l^2(\mathbb{R}))$.
But C_f is not locally bounded.
Why?

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$C_f : bv_1(l^2(\mathbb{R})) \to bv_1(l^2(\mathbb{R}))$.
But C_f is not locally bounded.
Why?f does not satisfy the Lipschitz condition on bounded subsets of $l^2(\mathbb{R})$.

Acting conditions between bv_p-spaces

Definition

A map $f : E \to E$ is called locally Hölder continuous in the stronger sense with exponent α, if there exist $\delta > 0$ and $L \geq 0$ such that $\|f(u) - f(w)\| \leq L \|u - w\|^{\alpha}$ for all $u, w \in E$ with $\|u - w\| \leq \delta$.

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Theorem

Let $1 < p \leq q < +\infty$. The composition operator C_f maps $bv_p(E)$ into $bv_q(E)$ if and only if f is locally Hölder continuous in the stronger sense with exponent $\frac{p}{q}$.

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Let $1 < p \leq q < +\infty$. The composition operator C_f maps $bv_p(E)$ into $bv_q(E)$ if and only if f is locally Hölder continuous in the stronger sense with exponent $\frac{p}{q}$.

Remark

In this case acting conditions ensure that C_f is locally bounded.

Pointwise continuity (between $bv_p(E)$)

Let $p, q \in [1, +\infty)$.

Theorem

Let E be a Banach space when $p = 1$, or a normed space when $p > 1$. Every composition operator C_f that maps $bv_p(E)$ into $bv_q(E)$ is continuous.

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Let E be a Banach space when $p = 1$, or a normed space when $p > 1$. Every composition operator C_f that maps $bv_p(E)$ into $bv_q(E)$ is continuous.

- If $1 \leq q < p < +\infty$, then C_f acts between $bv_p(E)$ and $bv_q(E)$ if and only if f is a constant map.

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- If $1 \leq q < p < +\infty$, then C_f acts between $bv_p(E)$ and $bv_q(E)$ if and only if f is a constant map.
- C_f maps $bv_1(E)$ into $bv_p(E)$ if and only if f is Hölder continuous on compact subsets of E with exponent $\frac{1}{p}$.

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- C_f maps $bv_1(E)$ into $bv_p(E)$ if and only if f is Hölder continuous on compact subsets of E with exponent $\frac{1}{p}$.
- If $1 \leq p < q < +\infty$. C_f maps $bv_p(E)$ into $bv_q(E)$ if and only if f is locally Hölder continuous in the stronger sense with exponent $\frac{p}{q}$.

Composition operators acting from bv_p-spaces

Theorem

Let $p, q \in [1, +\infty)$ and let $X \in \{c_0(E), l^q(E)\}$. The composition operator C_f maps $bv_p(E)$ into X if and only if its generator f is the zero function.

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Theorem

The composition operator C_f maps $bv_p(E)$ into $l^\infty(E)$ if and only if

1. *f is bounded on E for $p > 1$,*
2. *f is bounded on compact subsets of E, where E is a Banach space and $p = 1$.*

Composition operators acting from bv_p-spaces to $c(E)$

Theorem

Let E be a Banach space. Then, the following conditions are equivalent:

1. *the composition operator C_f maps $bv_1(E)$ into $c(E)$,*
2. *f is continuous on E,*
3. *f is continuous on bounded subsets of E,*
4. *f is continuous on compact subsets of E,*
5. *f is continuous on countable subsets of E.*

Composition operators acting from bv_p-spaces to $c(E)$

For $p > 1$ (Lipschitz) continuous mappings, in general, do not generate composition operators that act from $bv_p(E)$ into $c(E)$. To see this it suffices to consider the identity mapping on E, that is, $f\colon E \to E$ given by $f(u) = u$. It generates the identity composition operator C_f that maps $bv_p(E)$ onto $bv_p(E)$. But $bv_p(E) \not\subseteq c(E)$. And so, $C_f(bv_p(E)) \not\subseteq c(E)$.

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Theorem

Let $p \in (1, +\infty)$. The composition operator C_f maps $bv_p(E)$ into $c(E)$ if and only if f is a constant mapping.

Pointwise continuity (from $bv_1(E)$)

Theorem

Let E be a Banach space and let $X \in \{c(E), l^\infty(E)\}$. Moreover, assume that the composition operator C_f maps $bv_1(E)$ into X. Then, C_f is continuous if and only if f is.

Pointwise continuity (from $bv_1(E)$)

Theorem

Let E be a Banach space and let $X \in \{c(E), l^\infty(E)\}$. Moreover, assume that the composition operator C_f maps $bv_1(E)$ into X. Then, C_f is continuous if and only if f is.

Corollary

Let E be a Banach space. Every composition operator C_f that maps $bv_1(E)$ into $c(E)$ is continuous.

Pointwise continuity (from $bv_p(E)$)

Theorem

Let $X \in \{c(E), l^\infty(E)\}$ and assume that C_f maps $bv_p(E)$ into X. Then, C_f is continuous if and only if f is a constant mapping.

Local uniform continuity (with target space $bv_p(E)$)

Theorem

Assume that the composition operator C_f maps $c_0(E)$ into $bv_p(E)$. Then, C_f is locally uniformly continuous if and only if f is a constant map.

Local uniform continuity (with target space $bv_p(E)$)

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Assume that the composition operator C_f maps $c_0(E)$ into $bv_p(E)$. Then, C_f is locally uniformly continuous if and only if f is a constant map.

Theorem

Assume that C_f maps $l^p(E)$ into $bv_q(E)$. Then, it is locally uniformly continuous if and only if for every $\varepsilon > 0$ and $r > 0$ there exist $\delta > 0$ and constants $L_1, L_2, L_3 \geq 0$ satisfying $(L_1^q + L_2^q)r^p \leq \varepsilon^q$ such that for all $u, w \in B_E(0, r)$ we have

$$\|f(u) - f(w)\| \leq L_1 \|u\|^{\frac{p}{q}} + L_2 \|w\|^{\frac{p}{q}} + L_3 \|u - w\|^{\frac{p}{q}},$$

whenever $\|u - w\| \leq \delta$.

Local uniform continuity (between $bv_1(E)$)

Theorem

Let E be a Banach space. If the generator $f : E \to E$ is Fréchet differentiable with the derivative $f' \colon E \to \mathcal{L}(E)$ being locally uniformly continuous and locally bounded, then the composition operator C_f maps $bv_1(E)$ into itself and is locally uniformly continuous.

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Let E be a Banach space. If the generator $f : E \to E$ is Fréchet differentiable with the derivative $f' \colon E \to \mathcal{L}(E)$ being locally uniformly continuous and locally bounded, then the composition operator C_f maps $bv_1(E)$ into itself and is locally uniformly continuous.

Theorem

A composition operator C_f mapping $bv_1(\mathbb{R})$ into itself is locally uniformly continuous if and only if its generator f is continuously differentiable.

Local uniform continuity (between $bv_p(\mathbb{R})$)

Theorem

The composition operator C_f maps $bv_1(\mathbb{R})$ into $bv_p(\mathbb{R})$ and is locally uniformly continuous, if the function $f : \mathbb{R} \to \mathbb{R}$ satisfies one of the following conditions:

1. *f is continuously differentiable,*
2. *f is Hölder continuous on bounded subsets of $\mathbb{R}$ with exponent α such that $\frac{1}{p} < \alpha \leq 1$.*

Local uniform continuity (from $bv_1(\mathbb{E})$)

Theorem

Let E be a Banach space. Assume that the composition operator C_f maps:
- $bv_1(E)$ into $c(E)$ or
-$bv_1(E)$ into $l^\infty(E)$.
Then, C_f is locally uniformly continuous if and only if f is.

Local uniform continuity (between $bv_p(E)$)

$C_f : bv_p(E) \to bv_q(E)$ for $1 < p \leq q < +\infty$

???

Uniform continuity (between $bv_p(E)$)

Let $1 \leq p \leq q < +\infty$ and let E be a Banach space when $p = 1$, and a normed space when $p > 1$.

Theorem

Assume that C_f maps $bv_p(E)$ into $bv_q(E)$. Then, the following conditions are equivalent:

1. *C_f is Lipschitz continuous,*
2. *C_f is uniformly continuous,*
3. *the map $\varphi(u) := f(u) - f(0)$ is a continuous linear endomorphism of E.*

Uniform continuity (between $bv_p(\mathbb{R})$)

Corollary

C_f maps $bv_p(\mathbb{R})$ into $bv_q(\mathbb{R})$ is uniformly continuous (or, equivalently, Lipschitz continuous) if and only if its generator $f : \mathbb{R} \to \mathbb{R}$ is of the form $f(u) = au + b$ for some $a, b \in \mathbb{R}$.

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Thank you for your attention!