Various approaches to hyperconvexity in partial metric spaces

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Definition (Matthews)

Let U be a nonempty set. The mapping $p : U \times U \to \mathbb{R}_0^+$ is said to be a *partial metric* or *pmetric* on U if for any $x, y, z \in U$ the following four conditions hold:

(P1) $x = y$ if and only if $p(x,x) = p(y,y) = p(x,y)$;
(P2) $p(x,x) \leqslant p(x,y)$;
(P3) $p(x,y) = p(y,x)$;
(P4) $p(x,y) \leqslant p(x,z) + p(z,y) - p(z,z)$.

The set U with the partial metric p is called a *partial metric space*, which is denoted by (U,p). We call $p(x,x)$ the *size* of x.

Partial metric spaces

Definition

For a partial metric space (U, p) we define mappings $p^m : U \times U \to \mathbb{R}_0^+$, $d_m : U \times U \to \mathbb{R}_0^+$ and $D\colon U \times U \to \mathbb{R}_0^+$ by the formulae

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$$d_m(x, y) = max\{p(x, y) - p(x, x), p(x, y) - p(y, y)\}$$

for any $x, y \in U$,

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for any $x, y \in U$, and

$$D(x, y) := \begin{cases} p(x, y), & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

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Then p^m, d_m and D are metrics on U.

Notation

Let (U, p) be a partial metric space. Since for $\epsilon > 0$ and $x \in U$ the set $\{y \mid p(x, y) < \epsilon,\ y \in U\}$ may be empty, we define the open ball centered at $x \in U$ with radius $\epsilon > 0$ in the following way

$$B^p_\epsilon(x) := \{y \mid p(x, y) < p(x, x) + \epsilon,\ y \in U\}.$$

Topology of partial metric spaces

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Let (U, p) be a partial metric space. Since for $\epsilon > 0$ and $x \in U$ the set $\{y \mid p(x, y) < \epsilon,\ y \in U\}$ may be empty, we define the open ball centered at $x \in U$ with radius $\epsilon > 0$ in the following way

$$B_\epsilon^p(x) := \{y \mid p(x, y) < p(x, x) + \epsilon,\ y \in U\}.$$

The set of all open balls of the partial metric space (U, p) is the basis of a topology on U, denoted by $\mathcal{T}[p]$.

Completeness

Definition

A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of a partial metric space (U,p) is said to *converge to an element* $x \in U$ if

$$\lim_{n\to\infty} p(x_n,x) = p(x,x).$$

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A sequence $(x_n)_{n\in\mathbb{N}}$ in (U, p), is said to *properly converge* to $x \in U$, if it converges to x and $\lim_{n\to\infty} p(x_n, x_n) = p(x, x)$.

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A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of a partial metric space (U,p) is said to be a *Cauchy sequence* if there exists (and is finite)

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A partial metric space (U,p) is said to be *complete* if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (U,p) of elements of U is properly convergent.

Definition

For any $a \geqslant 0$, a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of a partial metric space (U, p) is said to be a *a-Cauchy sequence* if

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Similarly, a partial metric space (U, p) is said to be *a-complete* if every a-Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (U, p) of elements of U is properly convergent to some element $x \in U$.

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1. If a partial metric space (U, p) is complete then it is 0-complete. The converse is not true.
2. If a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of U is convergent to $x \in U$, then we can not deduce that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.
3. It turns out that a contractive mapping with respect to p may not be continuous with respect to the topology $\mathcal{T}[p]$.

Hyperconvexity

Definition (Aronszajn, Panitchpakdi)

A metric space (H, d) is said to be *hyperconvex* if for any family of closed balls $\{\overline{B}_d(x_i, r_i)\}_{i \in I}$ in H satisfying the condition $d(x_i, x_j) \leqslant r_i + r_j$ for any $i, j \in I$, there exists $x \in H$ with $d(x, x_i) \leqslant r_i$ for all $i \in I$.

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SOME CARE IS REQUIRED, when one attempts to reformulate the notion of hyperconvexity from the metric to the partial metric setting.

Proposition

Let (U, p) be a partial metric space consisting of at least two points. Assume that for any pair of closed balls $\overline{B}_p(x, r)$ and $\overline{B}_p(y, R)$ whose centers satisfy the condition $p(x, y) \leqslant r + R$, there exists a point $z \in U$ such that $p(x, z) \leqslant r$ and $p(z, y) \leqslant R$. Then, p is, in fact, a metric.

Remark

Hyperconvexity of a metric space means that for any family of closed balls $\{\overline{B}_d(x_i, r_i)\}_{i\in I}$ whose centers are not too far apart, there exists a point x lying in every ball $\overline{B}_d(x_i, r_i)$ centered at x_i. There is, however, another way of reasoning. The non-emptiness of $\bigcap_{i\in I} \overline{B}_d(x_i, r_i)$ also means that there exists a point y for which each x_i lies in the ball $\overline{B}_d(y, r_i)$.

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Definition

A partial metric space (U, p) is said to be *hyperconvex in the sense of Aronszajn and Panitchpakdi* (or, simply, *AP-hyperconvex*) if for any family of closed balls $\{\overline{B}_p(x_i, r_i)\}_{i\in I}$ in U satisfying the condition $p(x_i, x_j) \leqslant r_i + r_j$ for all $i, j \in I$, there exists a point $x \in U$ such that

$$p(x, x_i) \leqslant p(x_i, x_i) + r_i \text{ for every } i \in I. \tag{1}$$

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A partial metric space (U, p) is said to be *nodally hyperconvex* if for any family of closed balls $\{\overline{B}_p(x_i, r_i)\}_{i \in I}$ in U satisfying the condition $p(x_i, x_j) \leqslant r_i + r_j$ for all $i, j \in I$, there exists a point $x \in U$ such that

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Remark

1. The notions of hyperconvexity in partial metric spaces proposed above differ significantly.
2. Recall that every hyperconvex metric space (H, d) is totally convex, that is, for any points $x, y \in H$ and any non-negative numbers λ, μ such that $\lambda + \mu = d(x, y)$ there exists a point $z \in H$ such that $d(x, z) = \lambda$ and $d(z, y) = \mu$. For a partial metric space (U, p), total convexity may be introduced in two natural ways.

Hyperconvexity

One approach is to replace d with p in the original definition. Another is to declare (U, p) totally convex when its associated metric space (U, p^m) (or (U, d_m)) is totally convex in the classical sense. It can be shown that partial metric spaces that are hyperconvex in either of the above senses need not be totally convex.

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The following two propositions show a way to construct new hyperconvex partial metric spaces from existing ones.

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The following two propositions show a way to construct new hyperconvex partial metric spaces from existing ones.

Proposition

Let (U, p) be a bounded partial metric space which is AP-hyperconvex, and let a be a point that does not belong to U. Then, the set $W := U \cup \{a\}$ admits an AP-hyperconvex partial metric q that agrees with p on $U \times U$.

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Proposition

Let (U, p) be a bounded partial metric space, and let a be a point that does not belong to U. Then, the set $W := U \cup \{a\}$ admits a nodally hyperconvex partial metric q that agrees with p on $U \times U$.

Corollary

Every finite set admits a partial metric that is both AP- and nodally hyperconvex.

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Let (U, p) be a bounded partial metric space, and let a be a point that does not belong to U. Then, the set $W := U \cup \{a\}$ admits a nodally hyperconvex partial metric q that agrees with p on $U \times U$.

Corollary

Every finite set admits a partial metric that is both AP- and nodally hyperconvex.

Remark

AP-hyperconvex partial metric spaces as well as nodally hyperconvex partial metric spaces – unlike their classical metric counterparts – are not necessarily complete.

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Any complex Banach space can be endowed with a partial metric under which it is AP-hyperconvex. This stands in a contrast to the metric case.

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Definition

A partial metric space (U, p) is called *p^m-hyperconvex* (respectively, *d_m-hyperconvex*, or *D-hyperconvex*) when the associated metric space (U, p^m) (respectively, (U, d_m), or (U, D)) is hyperconvex in the classical sense.

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A partial metric space (U, p) is called *p^m-hyperconvex* (respectively, *d_m-hyperconvex*, or *D-hyperconvex*) when the associated metric space (U, p^m) (respectively, (U, d_m), or (U, D)) is hyperconvex in the classical sense.

Remark

If $(U; p)$ is a partial metric space with at least two points which is additionally D-hyperconvex, then p is a metric.

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Let (U, p) be a partial metric space. If U is p^m-hyperconvex, then it is also nodally hyperconvex.

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Proposition

Let (U, p) be a partial metric space. If U is d_m-hyperconvex, then it is both AP- and nodally hyperconvex.

Remark

Notice that the above two propositions are not reversible.

Remark

1. A partial metric that is p^m-hyperconvex need not be d_m-hyperconvex.

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1. A partial metric that is p^m-hyperconvex need not be d_m-hyperconvex.
2. One can show that in the case of bounded AP- and nodally hyperconvex partial metric spaces non-expansive mappings do not necessarily possess the fixed point property. Thus, another key feature of hyperconvex metric spaces can be lost when moving to the partial metric setting.

Open problems

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1. How to properly define a Lipschitz continuous mapping in a partial metric space (U, p)?
2. How to properly define the so called hyperconvex hull in a partial metric space (U, p)?

References

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Thank you

for

attention