Descriptive set theory in Banach spaces I

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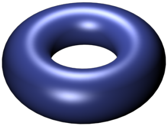
Summer School 2026: Topics in Banach Space Theory
Castro Urdiales

PART ONE: Classification problems and Descriptive Set Theory

Motivation. In mathematics we often try to classify certain objects.

- Vector spaces are isomorphic iff they have the same dimension
- Compact orientable surfaces are homeomorphic iff they have the same genus



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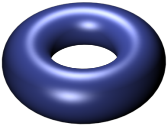

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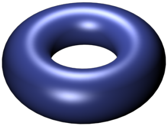

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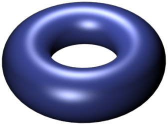

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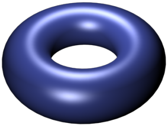

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A **Polish space** is a separable and completely metrizable topological space.

Definition

Let X, Y be Polish spaces and let E, F equivalence relations on X, Y respectively. E is said to be *Borel reducible* to F, if there is a Borel mapping $f: X \to Y$ such that for every $a, b \in X$ we have $aEb \Leftrightarrow f(a)Ff(b)$. We denote it by

$$E \leq_B F.$$

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Example 2: $U(n) := \{U \in \mathbb{C}^{n \times n}: U \text{ is unitary matrix}\} \subset \mathbb{C}^{n\times n}$ is Polish space. Consider $E := \{(A, B) \in U(n) \times U(n) \colon \exists V \in U(n) : A = VBV^*\}$.The mapping assigning to a matrix its eigenvalues with their multiplicities

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Coding of compact/Polish/Banach spaces: X universal space (e.g. $X = [0,1]^{\mathbb{N}}$ for compact metric spaces, $X = C([0,1])$ for Polish/Banach spaces). Consider the space of all

$$\text{compact metric spaces } \mathcal{K}(X) := \{K \subset X \colon K \text{ is compact}\},$$
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Pick $P(X) \in \{\mathcal{K}(X), F(X), SB(X)\}$. There exists a Polish topology τ on $P(X)$ such that Borel sets of $(P(X), \tau)$ are generated by sets of the form

$$E^{+}(U) := \{Y \in P(X) \colon Y \cap U \neq \emptyset\}, \quad U \subset X \text{ open}.$$

(If $P(X) = \mathcal{K}(X)$, τ is the Vietoris topology, otherwise there is no natural choice of topology τ.)

Examples of equivalence relations:

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Map of classication problems:

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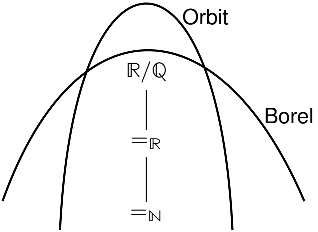

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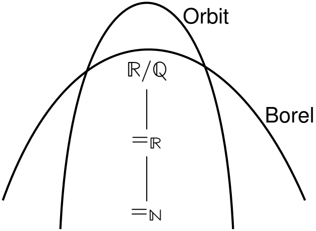

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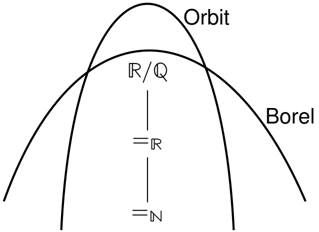

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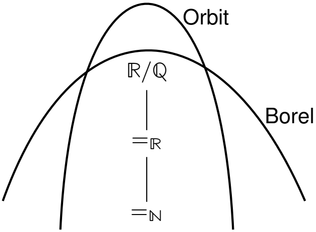

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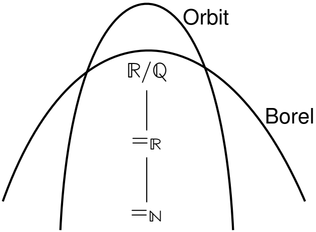

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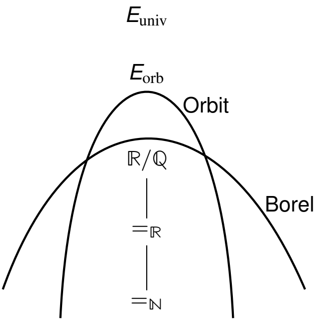

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E_{orb} = universal orbit equivalence,
E_{univ} = universal analytic equivalence.

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For example: $E_{\mathrm{Iso}} := \{(Y, X) \in SB(X)^2;\ \exists T \in \mathrm{Iso}(X) :\ TY = Z\}$

Note: $E_G \subset X^2$ is always analytic (= continuous image of Borel subset of Polish space), it might not be Borel.

Examples of classification problems

Classification problems Borel bireducible with $=_{\mathbb{R}}$:

- finite-dimensional Banach spaces up to isometry (folklore),
- compact metric spaces up to isometry (essentially Gromov).

Classification problems Borel bireducible with $\mathbb{R}/\mathbb{Q}$:

- arc-connection relation of points in Knaster continuum (Hevessy-Uyar-Vejnar, 2026).

Classification problems Borel bireducible with $E_{\mathbf{orb}}$:

- separable complete metric spaces up to isometry (Gao-Kechris, 2003),
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- metric compact spaces up to homeomorphism (Zielinski, 2016),
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PART TWO: Descriptive Set Theory in Banach spaces (overview)

$SB := SB(C([0,1])) = \{X \subset C([0,1]);\ X \text{ is closed subspace}\}$,
$SB_\infty := \{X \subset C([0,1]);\ X \text{ is } \infty\text{-dim closed subspace}\}$ .. class of separable infinite-dimensional Banach spaces.

1. Ergodic Banach spaces

Gowers-Maurey (1993): Let $X \in SB_\infty$ be isomorphic to all its subspaces (closed, ∞-dim). Then $X \simeq \ell_2$.

Question (Godefroy, Johnson): If $X \not\simeq \ell_2$, how many non-isomorphic SB_∞ subspaces can X have? Is it possible there are only two?

Definition (Ferenczi-Rosendal, 2005): $X \in SB_\infty$ is *ergodic* if $\mathbb{R}/\mathbb{Q}$ is Borel reducible to $\{(Z, Y) \in SB(X)^2;\ Y \simeq Z\}$.
(Ergodic Banach spaces have continuum many non-isomorphic subspaces, those cannot be classified by real numbers.)

Conjecture (Ferenczi-Rosendal, 2005): If $X \not\simeq \ell_2$ is SB_∞ space, then X is ergodic.

Some results supporting the conjecture:

- Anisca, 2010: If $X \not\simeq \ell_2$ is asymptotically Hilbertian, then it is ergodic.
  (X is asymptotically Hilbertian if
  $\exists K \geq 1\ \forall n \in \mathbb{N}\ \exists\, Y \subset X,\ \dim(X/Y) < \infty : \quad \forall E \subset Y,\ \dim E = n \Rightarrow E \cong_K \ell_2^n$.)
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It is well known that **WF** is not analytic.

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Theorem (Bourgain 80; Bossard 02)

Let $X \in SB_\infty$ be universal for all separable reflexive Banach spaces. Then X is universal for SB_∞.

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- *Antunes-Braga-Beanland 2021*: There exists a separable reflexive Banach space Z with a Schauder basis such that, whenever X is a separable Banach space isomorphic to $L_1[0,1]$ and $T\colon X \to Y$ is a weakly compact operator, then T factors through Z.

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3. Complexities of classes of Banach spaces

Bossard (1993): $\{X \in SB_\infty : X \text{ is not reflexive}\} \subset SB_\infty$ is not a Borel set

CDDK (2019): For every locally finite metric space M the set

$$\{X \in SB_\infty : M \text{ bi-Lipschitz embeds into } X\}$$

is Borel.

Application: There does not exist a locally finite metric space M such that any ∞-dim. sep. Banach space X is not reflexive iff M admits a bilipschitz embeddings into X.

Remark: For superreflexive spaces such M does exist
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"Problems" with coding SB_∞: not very pleasant to work with, does not distinguish between Borel classes (topology not naturally given)

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PART THREE: Complexities of classes of Banach spaces with emphasis on isometry classes (overview)

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