Descriptive set theory in Banach spaces II

Marek Cúth

Summer School 2026: Topics in Banach Space Theory
Castro Urdiales

Coding of Banach spaces: Let $V = c_{00}(\mathbb{Q})$ (finitely supported sequences of rational numbers).

$$\mathcal{B} = \{\mu \in [0,\infty)^V \colon \mu \text{ can be extended to a norm on } c_{00}\}.$$

Fact: $\mathcal{B} \subset \mathbb{R}^V$ is G_δ subset (thus, $\mathcal{B}$ is Polish)

We identify each $\mu \in \mathcal{B}$ with the Banach space $X_\mu := \overline{(c_{00}, \mu)}$.

Coding of Banach spaces: Let $V = c_{00}(\mathbb{Q})$ (finitely supported sequences of rational numbers).

$$\mathcal{B} = \{\mu \in [0,\infty)^V : \mu \text{ can be extended to a norm on } c_{00}\}.$$

Fact: $\mathcal{B} \subset \mathbb{R}^V$ is G_δ subset (thus, $\mathcal{B}$ is Polish)

We identify each $\mu \in \mathcal{B}$ with the Banach space $X_\mu := \overline{(c_{00}, \mu)}$.

Every separable infinite-dimensional Banach space is linearly isometric to X_μ for some $\mu \in \mathcal{B}$.

Coding of Banach spaces: Let $V = c_{00}(\mathbb{Q})$ (finitely supported sequences of rational numbers).

$$\mathcal{B} = \{\mu \in [0,\infty)^V \colon \mu \text{ can be extended to a norm on } c_{00}\}.$$

Fact: $\mathcal{B} \subset \mathbb{R}^V$ is G_δ subset (thus, $\mathcal{B}$ is Polish)

We identify each $\mu \in \mathcal{B}$ with the Banach space $X_\mu := \overline{(c_{00}, \mu)}$.

Every separable infinite-dimensional Banach space is linearly isometric to X_μ for some $\mu \in \mathcal{B}$.

Remark 1: Alternative approach suggested by Godefroy and Saint-Raymond 2018, they considered *admissible topologies* on SB_∞.

Recall from last lecture

Coding of Banach spaces: Let $V = c_{00}(\mathbb{Q})$ (finitely supported sequences of rational numbers).

$$\mathcal{B} = \{\mu \in [0,\infty)^V : \mu \text{ can be extended to a norm on } c_{00}\}.$$

Fact: $\mathcal{B} \subset \mathbb{R}^V$ is G_δ subset (thus, $\mathcal{B}$ is Polish)

We identify each $\mu \in \mathcal{B}$ with the Banach space $X_\mu := \overline{(c_{00}, \mu)}$.

Every separable infinite-dimensional Banach space is linearly isometric to X_μ for some $\mu \in \mathcal{B}$.

Remark 1: Alternative approach suggested by Godefroy and Saint-Raymond 2018, they considered *admissible topologies* on SB_∞.

Remark 2: Another alternative could be to use

$$\mathcal{P} = \{\mu \in [0,\infty)^V : \mu \text{ can be extended to a pseudonorm on } c_{00}\}$$

We identify each $\mu \in \mathcal{P}$ with the Banach space $X_\mu := \overline{(c_{00}, \mu)/_{\{x;\ \mu(x)=0\}}}$.

$$\mathcal{P}_\infty := \{\mu \in \mathcal{P};\ X_\mu \text{ is infinite-dimensional}\}$$

Fact: $\mathcal{P}$ and $\mathcal{P}_\infty$ are G_δ subsets of $\mathbb{R}^V$

Coding of Banach spaces: Let $V = c_{00}(\mathbb{Q})$ (finitely supported sequences of rational numbers).

$$\mathcal{B} = \{\mu \in [0,\infty)^V : \mu \text{ can be extended to a norm on } c_{00}\}.$$

Fact: $\mathcal{B} \subset \mathbb{R}^V$ is G_δ subset (thus, $\mathcal{B}$ is Polish)

We identify each $\mu \in \mathcal{B}$ with the Banach space $X_\mu := \overline{(c_{00}, \mu)}$.

Every separable infinite-dimensional Banach space is linearly isometric to X_μ for some $\mu \in \mathcal{B}$.

Remark 1: Alternative approach suggested by Godefroy and Saint-Raymond 2018, they considered *admissible topologies* on SB_∞.

Remark 2: Another alternative could be to use

$$\mathcal{P} = \{\mu \in [0,\infty)^V : \mu \text{ can be extended to a pseudonorm on } c_{00}\}$$

We identify each $\mu \in \mathcal{P}$ with the Banach space $X_\mu := \overline{(c_{00}, \mu)/_{\{x;\ \mu(x)=0\}}}$.

$$\mathcal{P}_\infty := \{\mu \in \mathcal{P};\ X_\mu \text{ is infinite-dimensional}\}$$

Fact: $\mathcal{P}$ and $\mathcal{P}_\infty$ are G_δ subsets of $\mathbb{R}^V$

Proposition: There is a F_σ-measurable mapping $\Phi : \mathcal{P}_\infty \to \mathcal{B}$ such that for every $\mu \in \mathcal{P}_\infty$ we have $X_\mu \equiv X_{\Phi(\mu)}$.

Complexities of classes of Banach spaces

Bossard (1993): $\{\mu \in \mathcal{B} \colon X_\mu \text{ is not reflexive}\} \subset \mathcal{B}$ is not a Borel set

CDDK (2019): For every locally finite metric space M the set

$$\{\mu \in \mathcal{B} \colon M \text{ bi-Lipschitz embeds into } X_\mu\}$$

is Borel.

Application: There does not exist a locally finite metric space M such that any ∞-dim. sep. Banach space X is not reflexive iff M admits a bilipschitz embeddings into X.

PART THREE: Complexities of classes of Banach spaces with emphasis on isometry classes (overview)

Recall: Our coding of Banach spaces (sep, ∞-dim) is

$$\mathcal{B} = \{\mu \in [0,\infty)^V \colon \mu \text{ can be extended to a norm on } c_{00}\}.$$

Given a ∞-dim sep Banach space X, we let

$$\langle X \rangle_\equiv := \{\mu \in \mathcal{B} \colon X_\mu \text{ is linearly isometric to } X\}.$$

- $\langle X \rangle_\equiv$ is Borel subset of $\mathcal{B}$ (for every ∞-dim sep X)

Recall: Our coding of Banach spaces (sep, ∞-dim) is

$$\mathcal{B} = \{\mu \in [0,\infty)^V \colon \mu \text{ can be extended to a norm on } c_{00}\}.$$

Given a ∞-dim sep Banach space X, we let

$$\langle X\rangle_{\equiv} := \{\mu \in \mathcal{B} \colon X_\mu \text{ is linearly isometric to } X\}.$$

- $\langle X\rangle_{\equiv}$ is Borel subset of $\mathcal{B}$ (for every ∞-dim sep X)

Complexity of classes of Banach spaces

Recall: Our coding of Banach spaces (sep, ∞-dim) is

$$\mathcal{B} = \{\mu \in [0,\infty)^V : \mu \text{ can be extended to a norm on } c_{00}\}.$$

Given a ∞-dim sep Banach space X, we let

$$\langle X \rangle_{\equiv} := \{\mu \in \mathcal{B} : X_\mu \text{ is linearly isometric to } X\}.$$

- $\langle X \rangle_{\equiv}$ is Borel subset of $\mathcal{B}$ (for every ∞-dim sep X)
- $\langle \ell_2 \rangle_{\equiv}$ is closed in $\mathcal{B}$

Recall: Our coding of Banach spaces (sep, ∞-dim) is

$$\mathcal{B} = \{\mu \in [0,\infty)^V \colon \mu \text{ can be extended to a norm on } c_{00}\}.$$

Given a ∞-dim sep Banach space X, we let

$$\langle X\rangle_\equiv := \{\mu \in \mathcal{B} \colon X_\mu \text{ is linearly isometric to } X\}.$$

- $\langle X\rangle_\equiv$ is Borel subset of $\mathcal{B}$ (for every ∞-dim sep X)
- $\langle \ell_2\rangle_\equiv$ is closed in $\mathcal{B}$

  Parallelogram law: $X \equiv \ell_2$ iff

  $$\forall x, y \in X : \quad \|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

  Thus,

  $$\langle \ell_2\rangle_\equiv = \bigcap_{x,y\in V} \{\mu \in \mathcal{B} \colon \mu(x+y)^2 + \mu(x-y)^2 = 2(\mu(x)^2 + \mu(y)^2)\}$$

Complexity of classes of Banach spaces

Recall: Our coding of Banach spaces (sep, ∞-dim) is

$$\mathcal{B} = \{\mu \in [0,\infty)^V \colon \mu \text{ can be extended to a norm on } c_{00}\}.$$

Given a ∞-dim sep Banach space X, we let

$$\langle X\rangle_{\equiv} := \{\mu \in \mathcal{B} \colon X_\mu \text{ is linearly isometric to } X\}.$$

- $\langle X\rangle_{\equiv}$ is Borel subset of $\mathcal{B}$ (for every ∞-dim sep X)
- $\langle \ell_2\rangle_{\equiv}$ is closed in $\mathcal{B}$

Recall: Our coding of Banach spaces (sep, ∞-dim) is

$$\mathcal{B} = \{\mu \in [0,\infty)^V : \mu \text{ can be extended to a norm on } c_{00}\}.$$

Given a ∞-dim sep Banach space X, we let

$$\langle X \rangle_{\equiv} := \{\mu \in \mathcal{B} \colon X_\mu \text{ is linearly isometric to } X\}.$$

- $\langle X \rangle_{\equiv}$ is Borel subset of $\mathcal{B}$ (for every ∞-dim sep X)
- $\langle \ell_2 \rangle_{\equiv}$ is closed in $\mathcal{B}$

Proposition

X separable infinite-dimensional, then

$$\overline{\langle X \rangle_{\equiv}} = \{\mu \in \mathcal{B} \colon X_\mu \textit{ finitely representable in } X\}.$$

Recall: Our coding of Banach spaces (sep, ∞-dim) is

$$\mathcal{B} = \{\mu \in [0,\infty)^V : \mu \text{ can be extended to a norm on } c_{00}\}.$$

Given a ∞-dim sep Banach space X, we let

$$\langle X\rangle_\equiv := \{\mu \in \mathcal{B} \colon X_\mu \text{ is linearly isometric to } X\}.$$

- $\langle X\rangle_\equiv$ is Borel subset of $\mathcal{B}$ (for every ∞-dim sep X)
- $\langle \ell_2\rangle_\equiv$ is closed in $\mathcal{B}$

Proposition

X separable infinite-dimensional, then

$$\overline{\langle X\rangle_\equiv} = \{\mu \in \mathcal{B} \colon X_\mu \textit{ finitely representable in } X\}.$$

- X is finitely representable in Y iff given $E \subset X$ finite-dim and $\varepsilon > 0$ there exists $F \subset Y$ which is $(1+\varepsilon)$-isomorphic to E.

Recall: Our coding of Banach spaces (sep, ∞-dim) is

$$\mathcal{B} = \{\mu \in [0,\infty)^V\colon \mu \text{ can be extended to a norm on } c_{00}\}.$$

Given a ∞-dim sep Banach space X, we let

$$\langle X\rangle_\equiv := \{\mu \in \mathcal{B}\colon X_\mu \text{ is linearly isometric to } X\}.$$

- $\langle X\rangle_\equiv$ is Borel subset of $\mathcal{B}$ (for every ∞-dim sep X)
- $\langle \ell_2\rangle_\equiv$ is closed in $\mathcal{B}$

Proposition

X separable infinite-dimensional, then

$$\overline{\langle X\rangle_\equiv} = \{\mu \in \mathcal{B}\colon X_\mu \textit{ finitely representable in } X\}.$$

- X is finitely representable in Y iff given $E \subset X$ finite-dim and $\varepsilon > 0$ there exists $F \subset Y$ which is $(1+\varepsilon)$-isomorphic to E.

Corollary 1: We always have $\langle \ell_2\rangle_\equiv \subset \overline{\langle X\rangle_\equiv}$.

Recall: Our coding of Banach spaces (sep, ∞-dim) is

$$\mathcal{B} = \{\mu \in [0,\infty)^V : \mu \text{ can be extended to a norm on } c_{00}\}.$$

Given a ∞-dim sep Banach space X, we let

$$\langle X \rangle_\equiv := \{\mu \in \mathcal{B} \colon X_\mu \text{ is linearly isometric to } X\}.$$

- $\langle X \rangle_\equiv$ is Borel subset of $\mathcal{B}$ (for every ∞-dim sep X)
- $\langle \ell_2 \rangle_\equiv$ is closed in $\mathcal{B}$

Proposition

X separable infinite-dimensional, then

$$\overline{\langle X \rangle_\equiv} = \{\mu \in \mathcal{B} \colon X_\mu \textit{ finitely representable in } X\}.$$

- X is finitely representable in Y iff given $E \subset X$ finite-dim and $\varepsilon > 0$ there exists $F \subset Y$ which is $(1+\varepsilon)$-isomorphic to E.

Corollary 1: We always have $\langle \ell_2 \rangle_\equiv \subset \overline{\langle X \rangle_\equiv}$. In particular, ℓ_2 is the only Banach space (sep, ∞-dim) whose isometry class is closed.

Complexity of classes of Banach spaces

Recall: Our coding of Banach spaces (sep, ∞-dim) is

$$\mathcal{B} = \{\mu \in [0,\infty)^V : \mu \text{ can be extended to a norm on } c_{00}\}.$$

Given a ∞-dim sep Banach space X, we let

$$\langle X \rangle_{\equiv} := \{\mu \in \mathcal{B} \colon X_\mu \text{ is linearly isometric to } X\}.$$

- $\langle X \rangle_{\equiv}$ is Borel subset of $\mathcal{B}$ (for every ∞-dim sep X)
- $\langle \ell_2 \rangle_{\equiv}$ is closed in $\mathcal{B}$

Proposition

X separable infinite-dimensional, then

$$\overline{\langle X \rangle_{\equiv}} = \{\mu \in \mathcal{B} \colon X_\mu \textit{ finitely representable in } X\}.$$

- X is finitely representable in Y iff given $E \subset X$ finite-dim and $\varepsilon > 0$ there exists $F \subset Y$ which is $(1+\varepsilon)$-isomorphic to E.

Corollary 1: We always have $\langle \ell_2 \rangle_{\equiv} \subset \overline{\langle X \rangle_{\equiv}}$. In particular, ℓ_2 is the only Banach space (sep, ∞-dim) whose isometry class is closed.

Corollary 2: c_0 finitely representable in X, then $\langle X \rangle_{\equiv}$ is dense in $\mathcal{B}$.

Complexity of classes of Banach spaces

Recall: Our coding of Banach spaces (sep, ∞-dim) is

$$\mathcal{B} = \{\mu \in [0,\infty)^V \colon \mu \text{ can be extended to a norm on } c_{00}\}.$$

Given a ∞-dim sep Banach space X, we let

$$\langle X \rangle_{\equiv} := \{\mu \in \mathcal{B} \colon X_\mu \text{ is linearly isometric to } X\}.$$

- $\langle X \rangle_{\equiv}$ is Borel subset of $\mathcal{B}$ (for every ∞-dim sep X)
- $\langle \ell_2 \rangle_{\equiv}$ is closed in $\mathcal{B}$

Proposition

X separable infinite-dimensional, then

$$\overline{\langle X \rangle_{\equiv}} = \{\mu \in \mathcal{B} \colon X_\mu \textit{ finitely representable in } X\}.$$

- X is finitely representable in Y iff given $E \subset X$ finite-dim and $\varepsilon > 0$ there exists $F \subset Y$ which is $(1+\varepsilon)$-isomorphic to E.

Corollary 1: We always have $\langle \ell_2 \rangle_{\equiv} \subset \overline{\langle X \rangle_{\equiv}}$. In particular, ℓ_2 is the only Banach space (sep, ∞-dim) whose isometry class is closed.

Corollary 2: c_0 finitely representable in X, then $\langle X \rangle_{\equiv}$ is dense in $\mathcal{B}$. In particular, there does not exist a separable Banach space with open isometry class.

Isometry classes of Banach spaces:

C-Doležal-Doucha-Kurka, 2024

- $\langle X \rangle_{\equiv}$ is Borel subset of $\mathcal{B}$ (for every ∞-dim sep X). It is never open.
- $\langle X \rangle_{\equiv}$ is closed iff $X \equiv \ell_2$
- $\langle \mathbb{G} \rangle_{\equiv}$ and $\langle L_p \rangle_{\equiv}$ are G_δ-complete for every $p \in [1, \infty)$, $p \neq 2$
- $\langle \ell_p \rangle_{\equiv}$ and $\langle c_0 \rangle_{\equiv}$ are $F_{\sigma\delta}$-complete

C-de Rancourt-Doucha, 2025-2026:

- $\langle X \rangle_{\equiv}$ is G_δ iff the Banach space is *guarded-Fraïssé*. Examples are:

$$\mathbb{G},\ L_p \text{ for } p \in [1, \infty),\ L_p(L_q) \text{ for } p, q \text{ such that } \ell_q \not\hookrightarrow L_p \text{ isometrically}$$

  (It is known that $\ell_q \hookrightarrow L_p$ isometrically iff $q \in \{2, p\}$ or $q \in (p, 2)$.)
- If $X \not\equiv \ell_2$ and $\langle X \rangle_{\equiv}$ is G_δ, then $\langle X \rangle_{\equiv}$ is G_δ-complete.

de Rancourt-Fakhoury, preprint from 2026:

- Certain Orlicz sequence spaces are guarded-Fraïssé, but do not contain ℓ_2 isometrically.

Isometry classes of Banach spaces:

C-Doležal-Doucha-Kurka, 2024

- $\langle X\rangle_{\equiv}$ is Borel subset of $\mathcal{B}$ (for every ∞-dim sep X). It is never open.
- $\langle X\rangle_{\equiv}$ is closed iff $X \equiv \ell_2$
- $\langle \mathbb{G}\rangle_{\equiv}$ and $\langle L_p\rangle_{\equiv}$ are G_δ-complete for every $p \in [1,\infty)$, $p \neq 2$
- $\langle \ell_p\rangle_{\equiv}$ and $\langle c_0\rangle_{\equiv}$ are $F_{\sigma\delta}$-complete

C-de Rancourt-Doucha, 2025-2026:

- $\langle X\rangle_{\equiv}$ is G_δ iff the Banach space is *guarded-Fraïssé*. Examples are:

$$\mathbb{G},\ \ L_p \text{ for } p \in [1,\infty),\ \ L_p(L_q) \text{ for } p,q \text{ such that } \ell_q \not\hookrightarrow L_p \text{ isometrically}$$

  (It is known that $\ell_q \hookrightarrow L_p$ isometrically iff $q \in \{2,p\}$ or $q \in (p,2)$.)
- If $X \not\equiv \ell_2$ and $\langle X\rangle_{\equiv}$ is G_δ, then $\langle X\rangle_{\equiv}$ is G_δ-complete.

de Rancourt-Fakhoury, preprint from 2026:

- Certain Orlicz sequence spaces are guarded-Fraïssé, but do not contain ℓ_2 isometrically.

Isometry classes of Banach spaces:

C-Doležal-Doucha-Kurka, 2024

- $\langle X\rangle_\equiv$ is Borel subset of $\mathcal{B}$ (for every ∞-dim sep X). It is never open.
- $\langle X\rangle_\equiv$ is closed iff $X \equiv \ell_2$
- $\langle \mathbb{G}\rangle_\equiv$ and $\langle L_p\rangle_\equiv$ are G_δ-complete for every $p \in [1,\infty)$, $p \neq 2$
- $\langle \ell_p\rangle_\equiv$ and $\langle c_0\rangle_\equiv$ are $F_{\sigma\delta}$-complete

C-de Rancourt-Doucha, 2025-2026:

- $\langle X\rangle_\equiv$ is G_δ iff the Banach space is *guarded-Fraïssé*. Examples are:

$$\mathbb{G},\ L_p \text{ for } p \in [1,\infty),\ L_p(L_q) \text{ for } p,q \text{ such that } \ell_q \not\hookrightarrow L_p \text{ isometrically}$$

  (It is known that $\ell_q \hookrightarrow L_p$ isometrically iff $q \in \{2,p\}$ or $q \in (p,2)$.)
- If $X \not\equiv \ell_2$ and $\langle X\rangle_\equiv$ is G_δ, then $\langle X\rangle_\equiv$ is G_δ-complete.

de Rancourt-Fakhoury, preprint from 2026:

- Certain Orlicz sequence spaces are guarded-Fraïssé, but do not contain ℓ_2 isometrically.

Isometry classes of Banach spaces:

C-Doležal-Doucha-Kurka, 2024

- $\langle X \rangle_\equiv$ is Borel subset of $\mathcal{B}$ (for every ∞-dim sep X). It is never open.
- $\langle X \rangle_\equiv$ is closed iff $X \equiv \ell_2$
- $\langle \mathbb{G} \rangle_\equiv$ and $\langle L_p \rangle_\equiv$ are G_δ-complete for every $p \in [1, \infty)$, $p \neq 2$
- $\langle \ell_p \rangle_\equiv$ and $\langle c_0 \rangle_\equiv$ are $F_{\sigma\delta}$-complete

C-de Rancourt-Doucha, 2025-2026:

- $\langle X \rangle_\equiv$ is G_δ iff the Banach space is *guarded-Fraïssé*. Examples are:

$$\mathbb{G},\ \ L_p \text{ for } p \in [1, \infty),\ \ L_p(L_q) \text{ for } p, q \text{ such that } \ell_q \not\hookrightarrow L_p \text{ isometrically}$$

  (It is known that $\ell_q \hookrightarrow L_p$ isometrically iff $q \in \{2, p\}$ or $q \in (p, 2)$.)
- If $X \not\equiv \ell_2$ and $\langle X \rangle_\equiv$ is G_δ, then $\langle X \rangle_\equiv$ is G_δ-complete.

de Rancourt-Fakhoury, preprint from 2026:

- Certain Orlicz sequence spaces are guarded-Fraïssé, but do not contain ℓ_2 isometrically.

Isometry classes of Banach spaces:

C-Doležal-Doucha-Kurka, 2024

- $\langle X\rangle_{\equiv}$ is Borel subset of $\mathcal{B}$ (for every ∞-dim sep X). It is never open.
- $\langle X\rangle_{\equiv}$ is closed iff $X \equiv \ell_2$
- $\langle\mathbb{G}\rangle_{\equiv}$ and $\langle L_p\rangle_{\equiv}$ are G_δ-complete for every $p \in [1,\infty)$, $p \neq 2$
- $\langle\ell_p\rangle_{\equiv}$ and $\langle c_0\rangle_{\equiv}$ are $F_{\sigma\delta}$-complete

C-de Rancourt-Doucha, 2025-2026:

- $\langle X\rangle_{\equiv}$ is G_δ iff the Banach space is *guarded-Fraïssé*. Examples are:

$$\mathbb{G},\ L_p \text{ for } p \in [1,\infty),\ L_p(L_q) \text{ for } p,q \text{ such that } \ell_q \not\hookrightarrow L_p \text{ isometrically}$$

(It is known that $\ell_q \hookrightarrow L_p$ isometrically iff $q \in \{2,p\}$ or $q \in (p,2)$.)

- If $X \not\equiv \ell_2$ and $\langle X\rangle_{\equiv}$ is G_δ, then $\langle X\rangle_{\equiv}$ is G_δ-complete.

de Rancourt-Fakhoury, preprint from 2026:

- Certain Orlicz sequence spaces are guarded-Fraïssé, but do not contain ℓ_2 isometrically.

Isometry classes of Banach spaces:

C-Doležal-Doucha-Kurka, 2024

- $\langle X\rangle_\equiv$ is Borel subset of $\mathcal{B}$ (for every ∞-dim sep X). It is never open.
- $\langle X\rangle_\equiv$ is closed iff $X \equiv \ell_2$
- $\langle \mathbb{G}\rangle_\equiv$ and $\langle L_p\rangle_\equiv$ are G_δ-complete for every $p \in [1,\infty)$, $p \neq 2$
- $\langle \ell_p\rangle_\equiv$ and $\langle c_0\rangle_\equiv$ are $F_{\sigma\delta}$-complete

C-de Rancourt-Doucha, 2025-2026:

- $\langle X\rangle_\equiv$ is G_δ iff the Banach space is *guarded-Fraïssé*. Examples are:

$$\mathbb{G},\ L_p \text{ for } p\in[1,\infty),\ L_p(L_q) \text{ for } p,q \text{ such that } \ell_q \not\hookrightarrow L_p \text{ isometrically}$$

  (It is known that $\ell_q \hookrightarrow L_p$ isometrically iff $q \in \{2,p\}$ or $q \in (p,2)$.)
- If $X \not\equiv \ell_2$ and $\langle X\rangle_\equiv$ is G_δ, then $\langle X\rangle_\equiv$ is G_δ-complete.

de Rancourt-Fakhoury, preprint from 2026:

- Certain Orlicz sequence spaces are guarded-Fraïssé, but do not contain ℓ_2 isometrically.

Isometry classes of Banach spaces:

C-Doležal-Doucha-Kurka, 2024

- $\langle X\rangle_\equiv$ is Borel subset of $\mathcal{B}$ (for every ∞-dim sep X). It is never open.
- $\langle X\rangle_\equiv$ is closed iff $X \equiv \ell_2$
- $\langle \mathbb{G}\rangle_\equiv$ and $\langle L_p\rangle_\equiv$ are G_δ-complete for every $p \in [1,\infty)$, $p \neq 2$
- $\langle \ell_p\rangle_\equiv$ and $\langle c_0\rangle_\equiv$ are $F_{\sigma\delta}$-complete

C-de Rancourt-Doucha, 2025-2026:

- $\langle X\rangle_\equiv$ is G_δ iff the Banach space is *guarded-Fraïssé*. Examples are:

$$\mathbb{G},\ L_p \text{ for } p \in [1,\infty),\ L_p(L_q) \text{ for } p,q \text{ such that } \ell_q \not\hookrightarrow L_p \text{ isometrically}$$

  (It is known that $\ell_q \hookrightarrow L_p$ isometrically iff $q \in \{2,p\}$ or $q \in (p,2)$.)
- If $X \not\equiv \ell_2$ and $\langle X\rangle_\equiv$ is G_δ, then $\langle X\rangle_\equiv$ is G_δ-complete.

de Rancourt-Fakhoury, preprint from 2026:

- Certain Orlicz sequence spaces are guarded-Fraïssé, but do not contain ℓ_2 isometrically.

Isomorphism classes of Banach spaces: it is very rarely the case that $\langle X\rangle_{\simeq}$ is Borel.

- $\langle X\rangle_{\simeq}$ is not Borel if X is one of the following spaces:

$$L_p \text{ for } p\in(1,\infty)\setminus\{2\},\quad c_0,\quad C(K) \text{ for any } K \text{ metrizable},$$
$$\text{reflexive Orlicz function spaces } L^{\Phi}[0,1] \text{ not isomorphic to } L^2[0,1]\ \ldots$$

  (*Bourgain 81; Bossard 02; Ghawadrah 16; Kurka 19*)
- $\langle \ell_p\rangle_{\simeq}$ is $\mathbf{\Pi}^0_{\omega+1}$ for $p\in(1,\infty)$ (**Open:** Is it of a finite Borel class?)

  (*Godefroy, Saint-Raymond 18*)
- $\langle X\rangle_{\simeq}$ is F_σ iff $X\simeq\ell_2$

  (*C, Doležal, Doucha, Kurka 24*),
- Spaces for which it is open whether $\langle X\rangle_{\simeq}$ is Borel:

$$\mathbb{G},\quad \ell_1$$

Several further estimates

- $\{\mu\in\mathcal{B};\ X_\mu \text{ has BAP}\}$ is $\mathbf{\Sigma}^0_6$ set (*Ghawadrah 20*)
- $\{\mu\in\mathcal{B};\ X_\mu \text{ is superreflexive}\}$ is $F_{\sigma,\delta}$ (*C, Doležal, Doucha, Kurka 24*)
- $\{\mu\in\mathcal{B};\ X_\mu \text{ has Daugavet property}\}$ is G_δ (*López-Pérez, Martínez Vañó, Rueda Zoca 26*)
- ...

Open: Is the following set Borel?

$$\{\mu\in\mathcal{B}:\ X_\mu \text{ is isometric to a } \mathcal{C}(K) \text{ space for some metric compact } K\}$$

Isomorphism classes of Banach spaces: it is very rarely the case that $\langle X\rangle_{\simeq}$ is Borel.

- $\langle X\rangle_{\simeq}$ is not Borel if X is one of the following spaces:

$$L_p \text{ for } p\in(1,\infty)\setminus\{2\},\quad c_0,\quad C(K) \text{ for any } K \text{ metrizable},$$
$$\text{reflexive Orlicz function spaces } L^{\Phi}[0,1] \text{ not isomorphic to } L^2[0,1]\ \dots$$

  (*Bourgain 81; Bossard 02; Ghawadrah 16; Kurka 19*)
- $\langle \ell_p\rangle_{\simeq}$ is $\mathbf{\Pi}^0_{\omega+1}$ for $p\in(1,\infty)$ (**Open:** Is it of a finite Borel class?)

  (*Godefroy, Saint-Raymond 18*)
- $\langle X\rangle_{\simeq}$ is F_σ iff $X\simeq\ell_2$

  (*C, Doležal, Doucha, Kurka 24*),
- Spaces for which it is open whether $\langle X\rangle_{\simeq}$ is Borel:

$$\mathbb{G},\quad \ell_1$$

Several further estimates

- $\{\mu\in\mathcal{B};\ X_\mu \text{ has BAP}\}$ is $\mathbf{\Sigma}^0_6$ set (*Ghawadrah 20*)
- $\{\mu\in\mathcal{B};\ X_\mu \text{ is superreflexive}\}$ is $F_{\sigma,\delta}$ (*C, Doležal, Doucha, Kurka 24*)
- $\{\mu\in\mathcal{B};\ X_\mu \text{ has Daugavet property}\}$ is G_δ (*López-Pérez, Martínez Vañó, Rueda Zoca 26*)
- ...

Open: Is the following set Borel?

$$\{\mu\in\mathcal{B}:\ X_\mu \text{ is isometric to a } \mathcal{C}(K) \text{ space for some metric compact } K\}$$

Isomorphism classes of Banach spaces: it is very rarely the case that $\langle X\rangle_{\simeq}$ is Borel.

- $\langle X\rangle_{\simeq}$ is not Borel if X is one of the following spaces:

$$L_p \text{ for } p\in(1,\infty)\setminus\{2\},\quad c_0,\quad C(K) \text{ for any } K \text{ metrizable},$$
$$\text{reflexive Orlicz function spaces } L^{\Phi}[0,1] \text{ not isomorphic to } L^2[0,1]\ \ldots$$

  (*Bourgain 81; Bossard 02; Ghawadrah 16; Kurka 19*)
- $\langle \ell_p\rangle_{\simeq}$ is $\mathbf{\Pi}^0_{\omega+1}$ for $p\in(1,\infty)$ (**Open:** Is it of a finite Borel class?)

  (*Godefroy, Saint-Raymond 18*)
- $\langle X\rangle_{\simeq}$ is F_σ iff $X\simeq\ell_2$

  (*C, Doležal, Doucha, Kurka 24*),
- Spaces for which it is open whether $\langle X\rangle_{\simeq}$ is Borel:

$$\mathbb{G},\quad \ell_1$$

Several further estimates

- $\{\mu\in\mathcal{B};\ X_\mu \text{ has BAP}\}$ is $\mathbf{\Sigma}^0_6$ set (*Ghawadrah 20*)
- $\{\mu\in\mathcal{B};\ X_\mu \text{ is superreflexive}\}$ is $F_{\sigma,\delta}$ (*C, Doležal, Doucha, Kurka 24*)
- $\{\mu\in\mathcal{B};\ X_\mu \text{ has Daugavet property}\}$ is G_δ (*López-Pérez, Martínez Vañó, Rueda Zoca 26*)
- ...

Open: Is the following set Borel?

$$\{\mu\in\mathcal{B}:\ X_\mu \text{ is isometric to a } \mathcal{C}(K) \text{ space for some metric compact } K\}$$

Isomorphism classes of separable Banach spaces - overview

Isomorphism classes of Banach spaces: it is very rarely the case that $\langle X\rangle_{\simeq}$ is Borel.

- $\langle X\rangle_{\simeq}$ is not Borel if X is one of the following spaces:

$$L_p \text{ for } p\in(1,\infty)\setminus\{2\},\quad c_0,\quad C(K) \text{ for any } K \text{ metrizable},$$
$$\text{reflexive Orlicz function spaces } L^{\Phi}[0,1] \text{ not isomorphic to } L^2[0,1]\ \ldots$$

  (*Bourgain 81; Bossard 02; Ghawadrah 16; Kurka 19*)
- $\langle \ell_p\rangle_{\simeq}$ is $\mathbf{\Pi}^0_{\omega+1}$ for $p\in(1,\infty)$ (**Open:** Is it of a finite Borel class?)

  (*Godefroy, Saint-Raymond 18*)
- $\langle X\rangle_{\simeq}$ is F_σ iff $X\simeq \ell_2$

  (*C, Doležal, Doucha, Kurka 24*),
- Spaces for which it is open whether $\langle X\rangle_{\simeq}$ is Borel:

$$\mathbb{G},\quad \ell_1$$

Several further estimates

- $\{\mu\in\mathcal{B};\ X_\mu \text{ has BAP}\}$ is $\mathbf{\Sigma}^0_6$ set (*Ghawadrah 20*)
- $\{\mu\in\mathcal{B};\ X_\mu \text{ is superreflexive}\}$ is $F_{\sigma,\delta}$ (*C, Doležal, Doucha, Kurka 24*)
- $\{\mu\in\mathcal{B};\ X_\mu \text{ has Daugavet property}\}$ is G_δ (*López-Pérez, Martínez Vañó, Rueda Zoca 26*)
- ...

Open: Is the following set Borel?

$$\{\mu\in\mathcal{B}:\ X_\mu \text{ is isometric to a } \mathcal{C}(K) \text{ space for some metric compact } K\}$$

Isomorphism classes of Banach spaces: it is very rarely the case that $\langle X\rangle_{\simeq}$ is Borel.

- $\langle X\rangle_{\simeq}$ is not Borel if X is one of the following spaces:

$$L_p \text{ for } p\in(1,\infty)\setminus\{2\},\quad c_0,\quad C(K) \text{ for any } K \text{ metrizable},$$
$$\text{reflexive Orlicz function spaces } L^{\Phi}[0,1] \text{ not isomorphic to } L^2[0,1]\ \ldots$$

  (*Bourgain 81; Bossard 02; Ghawadrah 16; Kurka 19*)
- $\langle \ell_p\rangle_{\simeq}$ is $\mathbf{\Pi}^0_{\omega+1}$ for $p\in(1,\infty)$ (**Open:** Is it of a finite Borel class?)

  (*Godefroy, Saint-Raymond 18*)
- $\langle X\rangle_{\simeq}$ is F_σ iff $X\simeq\ell_2$

  (*C, Doležal, Doucha, Kurka 24*),
- Spaces for which it is open whether $\langle X\rangle_{\simeq}$ is Borel:

$$\mathbb{G},\quad \ell_1$$

Several further estimates

- $\{\mu\in\mathcal{B};\ X_\mu \text{ has BAP}\}$ is $\mathbf{\Sigma}^0_6$ set (*Ghawadrah 20*)
- $\{\mu\in\mathcal{B};\ X_\mu \text{ is superreflexive}\}$ is $F_{\sigma,\delta}$ (*C, Doležal, Doucha, Kurka 24*)
- $\{\mu\in\mathcal{B};\ X_\mu \text{ has Daugavet property}\}$ is G_δ (*López-Pérez, Martínez Vañó, Rueda Zoca 26*)
- ...

Open: Is the following set Borel?

$$\{\mu\in\mathcal{B}:\ X_\mu \text{ is isometric to a } \mathcal{C}(K) \text{ space for some metric compact } K\}$$

Isomorphism classes of Banach spaces: it is very rarely the case that $\langle X\rangle_\simeq$ is Borel.

- $\langle X\rangle_\simeq$ is not Borel if X is one of the following spaces:

$$L_p \text{ for } p \in (1,\infty)\setminus\{2\}, \quad c_0, \quad C(K) \text{ for any } K \text{ metrizable},$$
$$\text{reflexive Orlicz function spaces } L^\Phi[0,1] \text{ not isomorphic to } L^2[0,1] \ \ldots$$

  (*Bourgain 81; Bossard 02; Ghawadrah 16; Kurka 19*)
- $\langle \ell_p\rangle_\simeq$ is $\mathbf{\Pi}^0_{\omega+1}$ for $p \in (1,\infty)$ (**Open:** Is it of a finite Borel class?)

  (*Godefroy, Saint-Raymond 18*)
- $\langle X\rangle_\simeq$ is F_σ iff $X \simeq \ell_2$

  (*C, Doležal, Doucha, Kurka 24*),
- Spaces for which it is open whether $\langle X\rangle_\simeq$ is Borel:

$$\mathbb{G}, \quad \ell_1$$

Several further estimates

- $\{\mu \in \mathcal{B};\ X_\mu \text{ has BAP}\}$ is $\mathbf{\Sigma}^0_6$ set (*Ghawadrah 20*)
- $\{\mu \in \mathcal{B};\ X_\mu \text{ is superreflexive}\}$ is $F_{\sigma,\delta}$ (*C, Doležal, Doucha, Kurka 24*)
- $\{\mu \in \mathcal{B};\ X_\mu \text{ has Daugavet property}\}$ is G_δ (*López-Pérez, Martínez Vañó, Rueda Zoca 26*)
- ...

Open: Is the following set Borel?

$$\{\mu \in \mathcal{B}:\ X_\mu \text{ is isometric to a } \mathcal{C}(K) \text{ space for some metric compact } K\}$$

PART FOUR: Isometry classes of $\mathcal{C}(K)$ spaces with K countable - upper bound for c
(based on joint work with M. Doležal, O. Kurka, J. Rondoš)

Cenzer-Mauldin 83: given $\beta = 0$ or β being a countable limit ordinal and $n \in \mathbb{N} \cup \{0\}$, the set of compact spaces with Cantor-Bendixson derivative less than $\beta + n$ is $\mathbf{\Sigma}^0_{\beta+n}$-complete.

Main aim: Identify Borel classes of $\langle \mathcal{C}(K) \rangle_{\equiv}$ for K countable.

Initial step: Identify Borel class of $\langle c \rangle_{\equiv}$.

Known:

- Any $X = \mathcal{C}(K)$ is L_1-predual (that is, $X^* = L_1(\mu)$ isometrically), being L_1-predual is G_δ condition.
- Given Banach space X and $\varepsilon > 0$, Szlenk derivative is given as

$$(B_{X^*})'_\varepsilon := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq \varepsilon\}.$$

  For every $\varepsilon \in (0,1)$: $X \equiv c_0$ iff X is L_1-predual and $(B_{X^*})'_{2\varepsilon} = (1-\varepsilon)B_{X^*}$. This can be shown to be an $F_{\sigma,\delta}$-condition.

First observation: Given $X = C([1,\omega])$ and $\varepsilon \in (0,1]$ we have

$$(B_{X^*})'_{2\varepsilon} = \{x^* \in B_{\ell_1([1,\omega])} : |x^*(\omega)| + (1 - \|x^*\|) \geq \varepsilon\}.$$

In particular, $(B_{X^*})'_2 = [-1,1]\delta_\omega$, so $(B_{X^*})'_2$ is a subset of a 1-dimensional space.

Having the property that "$(B_{X^*})'_2$ is a subset of a 1-dimensional space" is $F_{\sigma,\delta}$-condition.

Cenzer-Mauldin 83: given $\beta = 0$ or β being a countable limit ordinal and $n \in \mathbb{N} \cup \{0\}$, the set of compact spaces with Cantor-Bendixson derivative less than $\beta + n$ is $\mathbf{\Sigma}^0_{\beta+n}$-complete.

Main aim: Identify Borel classes of $\langle \mathcal{C}(K) \rangle_{\equiv}$ for K countable.

Initial step: Identify Borel class of $\langle c \rangle_{\equiv}$.

Known:

- Any $X = \mathcal{C}(K)$ is L_1-predual (that is, $X^* = L_1(\mu)$ isometrically), being L_1-predual is G_δ condition.
- Given Banach space X and $\varepsilon > 0$, Szlenk derivative is given as

$$(B_{X^*})'_\varepsilon := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq \varepsilon\}.$$

  For every $\varepsilon \in (0, 1)$: $X \equiv c_0$ iff X is L_1-predual and $(B_{X^*})'_{2\varepsilon} = (1 - \varepsilon)B_{X^*}$. This can be shown to be an $F_{\sigma,\delta}$-condition.

First observation: Given $X = C([1, \omega])$ and $\varepsilon \in (0, 1]$ we have

$$(B_{X^*})'_{2\varepsilon} = \{x^* \in B_{\ell_1([1,\omega])} : |x^*(\omega)| + (1 - \|x^*\|) \geq \varepsilon\}.$$

In particular, $(B_{X^*})'_2 = [-1, 1]\delta_\omega$, so $(B_{X^*})'_2$ is a subset of a 1-dimensional space.

Having the property that "$(B_{X^*})'_2$ is a subset of a 1-dimensional space" is $F_{\sigma,\delta}$-condition.

Cenzer-Mauldin 83: given $\beta = 0$ or β being a countable limit ordinal and $n \in \mathbb{N} \cup \{0\}$, the set of compact spaces with Cantor-Bendixson derivative less than $\beta + n$ is $\mathbf{\Sigma}^0_{\beta+n}$-complete.

Main aim: Identify Borel classes of $\langle \mathcal{C}(K) \rangle_\equiv$ for K countable.

Initial step: Identify Borel class of $\langle c \rangle_\equiv$.

Known:

- Any $X = \mathcal{C}(K)$ is L_1-predual (that is, $X^* = L_1(\mu)$ isometrically), being L_1-predual is G_δ condition.
- Given Banach space X and $\varepsilon > 0$, Szlenk derivative is given as

$$(B_{X^*})'_\varepsilon := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq \varepsilon\}.$$

  For every $\varepsilon \in (0,1)$: $X \equiv c_0$ iff X is L_1-predual and $(B_{X^*})'_{2\varepsilon} = (1-\varepsilon)B_{X^*}$. This can be shown to be an $F_{\sigma,\delta}$-condition.

First observation: Given $X = C([1,\omega])$ and $\varepsilon \in (0,1]$ we have

$$(B_{X^*})'_{2\varepsilon} = \{x^* \in B_{\ell_1([1,\omega])} : |x^*(\omega)| + (1 - \|x^*\|) \geq \varepsilon\}.$$

In particular, $(B_{X^*})'_2 = [-1,1]\delta_\omega$, so $(B_{X^*})'_2$ is a subset of a 1-dimensional space.

Having the property that "$(B_{X^*})'_2$ is a subset of a 1-dimensional space" is $F_{\sigma,\delta}$-condition.

Cenzer-Mauldin 83: given $\beta = 0$ or β being a countable limit ordinal and $n \in \mathbb{N} \cup \{0\}$, the set of compact spaces with Cantor-Bendixson derivative less than $\beta + n$ is $\mathbf{\Sigma}^0_{\beta+n}$-complete.

Main aim: Identify Borel classes of $\langle \mathcal{C}(K) \rangle_{\equiv}$ for K countable.

Initial step: Identify Borel class of $\langle c \rangle_{\equiv}$.

Known:

- Any $X = \mathcal{C}(K)$ is L_1-predual (that is, $X^* = L_1(\mu)$ isometrically), being L_1-predual is G_δ condition.
- Given Banach space X and $\varepsilon > 0$, Szlenk derivative is given as

$$(B_{X^*})'_\varepsilon := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq \varepsilon\}.$$

For every $\varepsilon \in (0,1)$: $X \equiv c_0$ iff X is L_1-predual and $(B_{X^*})'_{2\varepsilon} = (1-\varepsilon)B_{X^*}$. This can be shown to be an $F_{\sigma,\delta}$-condition.

First observation: Given $X = C([1,\omega])$ and $\varepsilon \in (0,1]$ we have

$$(B_{X^*})'_{2\varepsilon} = \{x^* \in B_{\ell_1([1,\omega])} : |x^*(\omega)| + (1 - \|x^*\|) \geq \varepsilon\}.$$

In particular, $(B_{X^*})'_2 = [-1,1]\delta_\omega$, so $(B_{X^*})'_2$ is a subset of a 1-dimensional space.

Having the property that "$(B_{X^*})'_2$ is a subset of a 1-dimensional space" is $F_{\sigma,\delta}$-condition.

Cenzer-Mauldin 83: given $\beta = 0$ or β being a countable limit ordinal and $n \in \mathbb{N} \cup \{0\}$, the set of compact spaces with Cantor-Bendixson derivative less than $\beta + n$ is $\mathbf{\Sigma}^0_{\beta+n}$-complete.

Main aim: Identify Borel classes of $\langle \mathcal{C}(K) \rangle_\equiv$ for K countable.

Initial step: Identify Borel class of $\langle c \rangle_\equiv$.

Known:

- Any $X = \mathcal{C}(K)$ is L_1-predual (that is, $X^* = L_1(\mu)$ isometrically), being L_1-predual is G_δ condition.
- Given Banach space X and $\varepsilon > 0$, Szlenk derivative is given as

$$(B_{X^*})'_\varepsilon := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq \varepsilon\}.$$

  For every $\varepsilon \in (0,1)$: $X \equiv c_0$ iff X is L_1-predual and $(B_{X^*})'_{2\varepsilon} = (1-\varepsilon)B_{X^*}$. This can be shown to be an $F_{\sigma,\delta}$-condition.

First observation: Given $X = C([1,\omega])$ and $\varepsilon \in (0,1]$ we have

$$(B_{X^*})'_{2\varepsilon} = \{x^* \in B_{\ell_1([1,\omega])} : |x^*(\omega)| + (1 - \|x^*\|) \geq \varepsilon\}.$$

In particular, $(B_{X^*})'_2 = [-1,1]\delta_\omega$, so $(B_{X^*})'_2$ is a subset of a 1-dimensional space.

Having the property that "$(B_{X^*})'_2$ is a subset of a 1-dimensional space" is $F_{\sigma,\delta}$-condition.

Inspiration by c_0

Cenzer-Mauldin 83: given $\beta = 0$ or β being a countable limit ordinal and $n \in \mathbb{N} \cup \{0\}$, the set of compact spaces with Cantor-Bendixson derivative less than $\beta + n$ is $\mathbf{\Sigma}^0_{\beta+n}$-complete.

Main aim: Identify Borel classes of $\langle \mathcal{C}(K) \rangle_{\equiv}$ for K countable.

Initial step: Identify Borel class of $\langle c \rangle_{\equiv}$.

Known:

- Any $X = \mathcal{C}(K)$ is L_1-predual (that is, $X^* = L_1(\mu)$ isometrically), being L_1-predual is G_δ condition.
- Given Banach space X and $\varepsilon > 0$, Szlenk derivative is given as

$$(B_{X^*})'_\varepsilon := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq \varepsilon\}.$$

For every $\varepsilon \in (0,1)$: $X \equiv c_0$ iff X is L_1-predual and $(B_{X^*})'_{2\varepsilon} = (1-\varepsilon)B_{X^*}$. This can be shown to be an $F_{\sigma,\delta}$-condition.

First observation: Given $X = C([1,\omega])$ and $\varepsilon \in (0,1]$ we have

$$(B_{X^*})'_{2\varepsilon} = \{x^* \in B_{\ell_1([1,\omega])} : |x^*(\omega)| + (1 - \|x^*\|) \geq \varepsilon\}.$$

In particular, $(B_{X^*})'_2 = [-1,1]\delta_\omega$, so $(B_{X^*})'_2$ is a subset of a 1-dimensional space.

Having the property that "$(B_{X^*})'_2$ is a subset of a 1-dimensional space" is $F_{\sigma,\delta}$-condition.

Cenzer-Mauldin 83: given $\beta = 0$ or β being a countable limit ordinal and $n \in \mathbb{N} \cup \{0\}$, the set of compact spaces with Cantor-Bendixson derivative less than $\beta + n$ is $\mathbf{\Sigma}^0_{\beta+n}$-complete.

Main aim: Identify Borel classes of $\langle \mathcal{C}(K) \rangle_{\equiv}$ for K countable.

Initial step: Identify Borel class of $\langle c \rangle_{\equiv}$.

Known:

- Any $X = \mathcal{C}(K)$ is L_1-predual (that is, $X^* = L_1(\mu)$ isometrically), being L_1-predual is G_δ condition.
- Given Banach space X and $\varepsilon > 0$, Szlenk derivative is given as

$$(B_{X^*})'_\varepsilon := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq \varepsilon\}.$$

For every $\varepsilon \in (0,1)$: $X \equiv c_0$ iff X is L_1-predual and $(B_{X^*})'_{2\varepsilon} = (1-\varepsilon)B_{X^*}$. This can be shown to be an $F_{\sigma,\delta}$-condition.

First observation: Given $X = C([1,\omega])$ and $\varepsilon \in (0,1]$ we have

$$(B_{X^*})'_{2\varepsilon} = \{x^* \in B_{\ell_1([1,\omega])} : |x^*(\omega)| + (1 - \|x^*\|) \geq \varepsilon\}.$$

In particular, $(B_{X^*})'_2 = [-1,1]\delta_\omega$, so $(B_{X^*})'_2$ is a subset of a 1-dimensional space.

Having the property that "$(B_{X^*})'_2$ is a subset of a 1-dimensional space" is $F_{\sigma,\delta}$-condition.

New ingredient needed

Recall:

$$(B_{X^*})'_\varepsilon := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq \varepsilon\}.$$

$\langle c_0 \rangle_\equiv$ is $F_{\sigma,\delta}$. We want to prove that $\langle c \rangle_\equiv$ is $F_{\sigma,\delta}$ (or something close to this).

- $X \equiv c_0$ iff X is L_1-predual and $(B_{X^*})'_1 = \frac{1}{2} B_{X^*}$. This is $F_{\sigma,\delta}$-condition.
- If $X \equiv c$, then

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

- Is $X \equiv c$ iff (*1) holds?
  No, it is not! It holds for example also for $X = c \oplus_\infty c_0$.

  Proof: We have $c \oplus_\infty c_0 \equiv C_0(K)$, where $K = [0,\omega) \uplus [0,\omega]$, that is, K is locally compact, not compact and all points of $K \setminus \{\omega\}$ are isolated.

  By the Riesz representation theorem, $C_0(K)^* = M(K) = \ell_1(K)$. We claim that $(B_{\ell_1(K)})'_2 \subset [-1,1]\delta_\omega$.

  Pick any $x^* \in \ell_1(K)$ for which there exists n_0 with $|x^*(n_0)| =: c > 0$. Consider the w^*-open neighborhood of x^* given as

$$U := \{z^* \in \ell_1(K);\ |z^*(n_0) - x^*(n_0)| < c/4\}.$$

  It suffices to show that $\operatorname{diam}(U \cap B_{\ell_1(K)}) < 2$. For every $z^* \in U \cap B_{\ell_1(K)}$ we have

$$\|z^*|_{K\setminus\{n_0\}}\|_1 \leq 1 - |z^*(n_0)| < 1 - |x^*(n_0)| + \tfrac{c}{4} < 1 - \tfrac{3}{4}c.$$

  Hence, for $z_1^*, z_2^* \in U \cap B_{\ell_1(K)}$ we have

$$\|z_1^* - z_2^*\|_1 \leq |(z_1^* - z_2^*)(n_0)| + 2(1 - \tfrac{3}{4}c) \leq \tfrac{c}{2} + 2(1 - \tfrac{3}{4}c) = 2 - c$$

New ingredient needed

Recall:

$$(B_{X^*})'_\varepsilon := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq \varepsilon\}.$$

$\langle c_0\rangle_\equiv$ is $F_{\sigma,\delta}$. We want to prove that $\langle c\rangle_\equiv$ is $F_{\sigma,\delta}$ (or something close to this).

- $X \equiv c_0$ iff X is L_1-predual and $(B_{X^*})'_1 = \frac{1}{2}B_{X^*}$. This is $F_{\sigma,\delta}$-condition.
- If $X \equiv c$, then

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

- Is $X \equiv c$ iff (*1) holds?

  No, it is not! It holds for example also for $X = c \oplus_\infty c_0$.

  Proof: We have $c \oplus_\infty c_0 \equiv C_0(K)$, where $K = [0,\omega) \uplus [0,\omega]$, that is, K is locally compact, not compact and all points of $K \setminus \{\omega\}$ are isolated.

  By the Riesz representation theorem, $C_0(K)^* = M(K) = \ell_1(K)$. We claim that $(B_{\ell_1(K)})'_2 \subset [-1,1]\delta_\omega$.

  Pick any $x^* \in \ell_1(K)$ for which there exists n_0 with $|x^*(n_0)| =: c > 0$. Consider the w^*-open neighborhood of x^* given as

$$U := \{z^* \in \ell_1(K);\ |z^*(n_0) - x^*(n_0)| < c/4\}.$$

  It suffices to show that $\operatorname{diam}(U \cap B_{\ell_1(K)}) < 2$. For every $z^* \in U \cap B_{\ell_1(K)}$ we have

$$\|z^*|_{K\setminus\{n_0\}}\|_1 \leq 1 - |z^*(n_0)| < 1 - |x^*(n_0)| + \tfrac{c}{4} < 1 - \tfrac{3}{4}c.$$

  Hence, for $z_1^*, z_2^* \in U \cap B_{\ell_1(K)}$ we have

$$\|z_1^* - z_2^*\|_1 \leq |(z_1^* - z_2^*)(n_0)| + 2(1 - \tfrac{3}{4}c) \leq \tfrac{c}{2} + 2(1 - \tfrac{3}{4}c) = 2 - c$$

New ingredient needed

Recall:

$$(B_{X^*})'_\varepsilon := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq \varepsilon\}.$$

$\langle c_0 \rangle_\equiv$ is $F_{\sigma,\delta}$. We want to prove that $\langle c \rangle_\equiv$ is $F_{\sigma,\delta}$ (or something close to this).

- $X \equiv c_0$ iff X is L_1-predual and $(B_{X^*})'_1 = \frac{1}{2} B_{X^*}$. This is $F_{\sigma,\delta}$-condition.
- If $X \equiv c$, then

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

- Is $X \equiv c$ iff (*1) holds?
  No, it is not! It holds for example also for $X = c \oplus_\infty c_0$.

  Proof: We have $c \oplus_\infty c_0 \equiv C_0(K)$, where $K = [0,\omega) \uplus [0,\omega]$, that is, K is locally compact, not compact and all points of $K \setminus \{\omega\}$ are isolated.

  By the Riesz representation theorem, $C_0(K)^* = M(K) = \ell_1(K)$. We claim that $(B_{\ell_1(K)})'_2 \subset [-1,1]\delta_\omega$.

  Pick any $x^* \in \ell_1(K)$ for which there exists n_0 with $|x^*(n_0)| =: c > 0$. Consider the w^*-open neighborhood of x^* given as

$$U := \{z^* \in \ell_1(K);\ |z^*(n_0) - x^*(n_0)| < c/4\}.$$

  It suffices to show that $\operatorname{diam}(U \cap B_{\ell_1(K)}) < 2$. For every $z^* \in U \cap B_{\ell_1(K)}$ we have

$$\|z^*|_{K\setminus\{n_0\}}\|_1 \leq 1 - |z^*(n_0)| < 1 - |x^*(n_0)| + \tfrac{c}{4} < 1 - \tfrac{3}{4}c.$$

  Hence, for $z_1^*, z_2^* \in U \cap B_{\ell_1(K)}$ we have

$$\|z_1^* - z_2^*\|_1 \leq |(z_1^* - z_2^*)(n_0)| + 2(1 - \tfrac{3}{4}c) \leq \tfrac{c}{2} + 2(1 - \tfrac{3}{4}c) = 2 - c$$

New ingredient needed

Recall:

$$(B_{X^*})'_\varepsilon := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq \varepsilon\}.$$

$\langle c_0\rangle_\equiv$ is $F_{\sigma,\delta}$. We want to prove that $\langle c\rangle_\equiv$ is $F_{\sigma,\delta}$ (or something close to this).

- $X \equiv c_0$ iff X is L_1-predual and $(B_{X^*})'_1 = \frac{1}{2}B_{X^*}$. This is $F_{\sigma,\delta}$-condition.
- If $X \equiv c$, then

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

- Is $X \equiv c$ iff (*1) holds?
  No, it is not! It holds for example also for $X = c \oplus_\infty c_0$.

  Proof: We have $c \oplus_\infty c_0 \equiv C_0(K)$, where $K = [0,\omega) \uplus [0,\omega]$, that is, K is locally compact, not compact and all points of $K \setminus \{\omega\}$ are isolated.

  By the Riesz representation theorem, $C_0(K)^* = M(K) = \ell_1(K)$. We claim that $(B_{\ell_1(K)})'_2 \subset [-1,1]\delta_\omega$.

  Pick any $x^* \in \ell_1(K)$ for which there exists n_0 with $|x^*(n_0)| =: c > 0$. Consider the w^*-open neighborhood of x^* given as

$$U := \{z^* \in \ell_1(K);\ |z^*(n_0) - x^*(n_0)| < c/4\}.$$

  It suffices to show that $\operatorname{diam}(U \cap B_{\ell_1(K)}) < 2$. For every $z^* \in U \cap B_{\ell_1(K)}$ we have

$$\|z^*|_{K\setminus\{n_0\}}\|_1 \leq 1 - |z^*(n_0)| < 1 - |x^*(n_0)| + \tfrac{c}{4} < 1 - \tfrac{3}{4}c.$$

  Hence, for $z_1^*, z_2^* \in U \cap B_{\ell_1(K)}$ we have

$$\|z_1^* - z_2^*\|_1 \leq |(z_1^* - z_2^*)(n_0)| + 2(1 - \tfrac{3}{4}c) \leq \tfrac{c}{2} + 2(1 - \tfrac{3}{4}c) = 2 - c$$

New ingredient needed

Recall:

$$(B_{X^*})'_\varepsilon := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq \varepsilon\}.$$

$\langle c_0\rangle_\equiv$ is $F_{\sigma,\delta}$. We want to prove that $\langle c\rangle_\equiv$ is $F_{\sigma,\delta}$ (or something close to this).

- $X \equiv c_0$ iff X is L_1-predual and $(B_{X^*})'_1 = \frac{1}{2}B_{X^*}$. This is $F_{\sigma,\delta}$-condition.
- If $X \equiv c$, then

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

- Is $X \equiv c$ iff (*1) holds?
  No, it is not! It holds for example also for $X = c \oplus_\infty c_0$.

  Proof: We have $c \oplus_\infty c_0 \equiv C_0(K)$, where $K = [0,\omega) \uplus [0,\omega]$, that is, K is locally compact, not compact and all points of $K \setminus \{\omega\}$ are isolated.

  By the Riesz representation theorem, $C_0(K)^* = M(K) = \ell_1(K)$. We claim that $(B_{\ell_1(K)})'_2 \subset [-1,1]\delta_\omega$.

  Pick any $x^* \in \ell_1(K)$ for which there exists n_0 with $|x^*(n_0)| =: c > 0$. Consider the w^*-open neighborhood of x^* given as

$$U := \{z^* \in \ell_1(K);\ |z^*(n_0) - x^*(n_0)| < c/4\}.$$

  It suffices to show that $\operatorname{diam}(U \cap B_{\ell_1(K)}) < 2$. For every $z^* \in U \cap B_{\ell_1(K)}$ we have

$$\|z^*|_{K\setminus\{n_0\}}\|_1 \leq 1 - |z^*(n_0)| < 1 - |x^*(n_0)| + \tfrac{c}{4} < 1 - \tfrac{3}{4}c.$$

  Hence, for $z_1^*, z_2^* \in U \cap B_{\ell_1(K)}$ we have

$$\|z_1^* - z_2^*\|_1 \leq |(z_1^* - z_2^*)(n_0)| + 2(1 - \tfrac{3}{4}c) \leq \tfrac{c}{2} + 2(1 - \tfrac{3}{4}c) = 2 - c$$

New ingredient needed

Recall:

$$(B_{X^*})'_\varepsilon := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq \varepsilon\}.$$

$\langle c_0 \rangle_\equiv$ is $F_{\sigma,\delta}$. We want to prove that $\langle c \rangle_\equiv$ is $F_{\sigma,\delta}$ (or something close to this).

- $X \equiv c_0$ iff X is L_1-predual and $(B_{X^*})'_1 = \frac{1}{2} B_{X^*}$. This is $F_{\sigma,\delta}$-condition.
- If $X \equiv c$, then

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

- Is $X \equiv c$ iff (*1) holds?
  No, it is not! It holds for example also for $X = c \oplus_\infty c_0$.

  Proof: We have $c \oplus_\infty c_0 \equiv C_0(K)$, where $K = [0,\omega) \uplus [0,\omega]$, that is, K is locally compact, not compact and all points of $K \setminus \{\omega\}$ are isolated.

  By the Riesz representation theorem, $C_0(K)^* = M(K) = \ell_1(K)$. We claim that $(B_{\ell_1(K)})'_2 \subset [-1,1]\delta_\omega$.

  Pick any $x^* \in \ell_1(K)$ for which there exists n_0 with $|x^*(n_0)| =: c > 0$. Consider the w^*-open neighborhood of x^* given as

$$U := \{z^* \in \ell_1(K);\ |z^*(n_0) - x^*(n_0)| < c/4\}.$$

  It suffices to show that $\operatorname{diam}(U \cap B_{\ell_1(K)}) < 2$. For every $z^* \in U \cap B_{\ell_1(K)}$ we have

$$\|z^*|_{K\setminus\{n_0\}}\|_1 \leq 1 - |z^*(n_0)| < 1 - |x^*(n_0)| + \tfrac{c}{4} < 1 - \tfrac{3}{4}c.$$

  Hence, for $z_1^*, z_2^* \in U \cap B_{\ell_1(K)}$ we have

$$\|z_1^* - z_2^*\|_1 \leq |(z_1^* - z_2^*)(n_0)| + 2(1 - \tfrac{3}{4}c) \leq \tfrac{c}{2} + 2(1 - \tfrac{3}{4}c) = 2 - c$$

New ingredient needed

Recall:

$$(B_{X^*})'_\varepsilon := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq \varepsilon\}.$$

$\langle c_0\rangle_\equiv$ is $F_{\sigma,\delta}$. We want to prove that $\langle c\rangle_\equiv$ is $F_{\sigma,\delta}$ (or something close to this).

- $X \equiv c_0$ iff X is L_1-predual and $(B_{X^*})'_1 = \frac{1}{2}B_{X^*}$. This is $F_{\sigma,\delta}$-condition.
- If $X \equiv c$, then

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \qquad (*1)$$

- Is $X \equiv c$ iff (*1) holds?
  No, it is not! It holds for example also for $X = c \oplus_\infty c_0$.

  Proof: We have $c \oplus_\infty c_0 \equiv C_0(K)$, where $K = [0,\omega) \uplus [0,\omega]$, that is, K is locally compact, not compact and all points of $K \setminus \{\omega\}$ are isolated.

  By the Riesz representation theorem, $C_0(K)^* = M(K) = \ell_1(K)$. We claim that $(B_{\ell_1(K)})'_2 \subset [-1,1]\delta_\omega$.

  Pick any $x^* \in \ell_1(K)$ for which there exists n_0 with $|x^*(n_0)| =: c > 0$. Consider the w^*-open neighborhood of x^* given as

$$U := \{z^* \in \ell_1(K);\ |z^*(n_0) - x^*(n_0)| < c/4\}.$$

  It suffices to show that $\operatorname{diam}(U \cap B_{\ell_1(K)}) < 2$. For every $z^* \in U \cap B_{\ell_1(K)}$ we have

$$\|z^*|_{K\setminus\{n_0\}}\|_1 \leq 1 - |z^*(n_0)| < 1 - |x^*(n_0)| + \tfrac{c}{4} < 1 - \tfrac{3}{4}c.$$

  Hence, for $z_1^*, z_2^* \in U \cap B_{\ell_1(K)}$ we have

$$\|z_1^* - z_2^*\|_1 \leq |(z_1^* - z_2^*)(n_0)| + 2(1 - \tfrac{3}{4}c) \leq \tfrac{c}{2} + 2(1 - \tfrac{3}{4}c) = 2 - c$$

New ingredient needed

Recall:

$$(B_{X^*})'_\varepsilon := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq \varepsilon\}.$$

$\langle c_0\rangle_\equiv$ is $F_{\sigma,\delta}$. We want to prove that $\langle c\rangle_\equiv$ is $F_{\sigma,\delta}$ (or something close to this).

- $X \equiv c_0$ iff X is L_1-predual and $(B_{X^*})'_1 = \frac{1}{2}B_{X^*}$. This is $F_{\sigma,\delta}$-condition.
- If $X \equiv c$, then

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

- Is $X \equiv c$ iff (*1) holds?
  No, it is not! It holds for example also for $X = c \oplus_\infty c_0$.

  Proof: We have $c \oplus_\infty c_0 \equiv C_0(K)$, where $K = [0,\omega) \uplus [0,\omega]$, that is, K is locally compact, not compact and all points of $K \setminus \{\omega\}$ are isolated.

  By the Riesz representation theorem, $C_0(K)^* = M(K) = \ell_1(K)$. We claim that $(B_{\ell_1(K)})'_2 \subset [-1,1]\delta_\omega$.

  Pick any $x^* \in \ell_1(K)$ for which there exists n_0 with $|x^*(n_0)| =: c > 0$. Consider the w^*-open neighborhood of x^* given as

$$U := \{z^* \in \ell_1(K);\ |z^*(n_0) - x^*(n_0)| < c/4\}.$$

  It suffices to show that $\operatorname{diam}(U \cap B_{\ell_1(K)}) < 2$. For every $z^* \in U \cap B_{\ell_1(K)}$ we have

$$\|z^*|_{K\setminus\{n_0\}}\|_1 \leq 1 - |z^*(n_0)| < 1 - |x^*(n_0)| + \frac{c}{4} < 1 - \frac{3}{4}c.$$

  Hence, for $z_1^*, z_2^* \in U \cap B_{\ell_1(K)}$ we have

$$\|z_1^* - z_2^*\|_1 \leq |(z_1^* - z_2^*)(n_0)| + 2(1 - \tfrac{3}{4}c) \leq \tfrac{c}{2} + 2(1 - \tfrac{3}{4}c) = 2 - c$$

New ingredient needed

Recall:

$$(B_{X^*})'_\varepsilon := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq \varepsilon\}.$$

$\langle c_0\rangle_\equiv$ is $F_{\sigma,\delta}$. We want to prove that $\langle c\rangle_\equiv$ is $F_{\sigma,\delta}$ (or something close to this).

- $X \equiv c_0$ iff X is L_1-predual and $(B_{X^*})'_1 = \frac{1}{2}B_{X^*}$. This is $F_{\sigma,\delta}$-condition.
- If $X \equiv c$, then

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

- Is $X \equiv c$ iff (*1) holds?
  No, it is not! It holds for example also for $X = c \oplus_\infty c_0$.

  Proof: We have $c \oplus_\infty c_0 \equiv C_0(K)$, where $K = [0,\omega) \uplus [0,\omega]$, that is, K is locally compact, not compact and all points of $K \setminus \{\omega\}$ are isolated.

  By the Riesz representation theorem, $C_0(K)^* = M(K) = \ell_1(K)$. We claim that $(B_{\ell_1(K)})'_2 \subset [-1,1]\delta_\omega$.

  Pick any $x^* \in \ell_1(K)$ for which there exists n_0 with $|x^*(n_0)| =: c > 0$. Consider the w^*-open neighborhood of x^* given as

$$U := \{z^* \in \ell_1(K);\ |z^*(n_0) - x^*(n_0)| < c/4\}.$$

  It suffices to show that $\operatorname{diam}(U \cap B_{\ell_1(K)}) < 2$. For every $z^* \in U \cap B_{\ell_1(K)}$ we have

$$\|z^*|_{K\setminus\{n_0\}}\|_1 \leq 1 - |z^*(n_0)| < 1 - |x^*(n_0)| + \tfrac{c}{4} < 1 - \tfrac{3}{4}c.$$

  Hence, for $z_1^*, z_2^* \in U \cap B_{\ell_1(K)}$ we have

$$\|z_1^* - z_2^*\|_1 \leq |(z_1^* - z_2^*)(n_0)| + 2(1 - \tfrac{3}{4}c) \leq \tfrac{c}{2} + 2(1 - \tfrac{3}{4}c) = 2 - c$$

Recall:

$$(B_{X^*})'_2 := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq 2\}.$$

If $X \equiv c$, then

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

Assume X satisfies (*1). Then $X^* \equiv \ell_1$.

Proof.

Known results

- If X is separable L_1-predual, then $X^* \not\equiv \ell_1$ iff $L_1[0,1] \subset X^*$ isometrically
- $L_1[0,1]$ has D2P (every weakly open subset of $B_{L_1[0,1]}$ has diameter two)

Assume $Z \subset X^*$ is isometric to $L_1[0,1]$. Pick U be a w^*-open subset of X^* such that $U \cap B_Z \neq \emptyset$. Then $U \cap Z$ is weakly open in Z, so $\operatorname{diam}(U \cap B_Z) = 2$. Hence, $B_Z \subset (B_{X^*})'_2$, contradiction with (*1). □

Theorem: Let K be a countable compact space with accumulation points K' and put $X = C(K)$. Then $(B_{X^*})'_2 = B_{M(K')}$.

Hence, if (*1) holds and moreover $X = \mathcal{C}(K)$ for some compact space K, then $X \equiv c$.

Recall:

$$(B_{X^*})'_2 := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq 2\}.$$

If $X \equiv c$, then

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

Assume X satisfies (*1). Then $X^* \equiv \ell_1$.

Proof.

Known results

- If X is separable L_1-predual, then $X^* \not\equiv \ell_1$ iff $L_1[0,1] \subset X^*$ isometrically
- $L_1[0,1]$ has D2P (every weakly open subset of $B_{L_1[0,1]}$ has diameter two)

Assume $Z \subset X^*$ is isometric to $L_1[0,1]$. Pick U be a w^*-open subset of X^* such that $U \cap B_Z \neq \varnothing$. Then $U \cap Z$ is weakly open in Z, so $\operatorname{diam}(U \cap B_Z) = 2$. Hence, $B_Z \subset (B_{X^*})'_2$, contradiction with (*1). □

Theorem: Let K be a countable compact space with accumulation points K' and put $X = C(K)$. Then $(B_{X^*})'_2 = B_{M(K')}$.

Hence, if (*1) holds and moreover $X = \mathcal{C}(K)$ for some compact space K, then $X \equiv c$.

Recall:

$$(B_{X^*})'_2 := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq 2\}.$$

If $X \equiv c$, then

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

Assume X satisfies (*1). Then $X^* \equiv \ell_1$.

Proof.

Known results

- If X is separable L_1-predual, then $X^* \not\equiv \ell_1$ iff $L_1[0,1] \subset X^*$ isometrically
- $L_1[0,1]$ has D2P (every weakly open subset of $B_{L_1[0,1]}$ has diameter two)

Assume $Z \subset X^*$ is isometric to $L_1[0,1]$. Pick U be a w^*-open subset of X^* such that $U \cap B_Z \neq \varnothing$. Then $U \cap Z$ is weakly open in Z, so $\operatorname{diam}(U \cap B_Z) = 2$. Hence, $B_Z \subset (B_{X^*})'_2$, contradiction with (*1). □

Theorem: Let K be a countable compact space with accumulation points K' and put $X = C(K)$. Then $(B_{X^*})'_2 = B_{M(K')}$.

Hence, if (*1) holds and moreover $X = \mathcal{C}(K)$ for some compact space K, then $X \equiv c$.

Recall:

$$(B_{X^*})'_2 := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq 2\}.$$

If $X \equiv c$, then

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

Assume X satisfies (*1). Then $X^* \equiv \ell_1$.

Proof.

Known results

- If X is separable L_1-predual, then $X^* \not\equiv \ell_1$ iff $L_1[0,1] \subset X^*$ isometrically
- $L_1[0,1]$ has D2P (every weakly open subset of $B_{L_1[0,1]}$ has diameter two)

Assume $Z \subset X^*$ is isometric to $L_1[0,1]$. Pick U be a w^*-open subset of X^* such that $U \cap B_Z \neq \varnothing$. Then $U \cap Z$ is weakly open in Z, so $\operatorname{diam}(U \cap B_Z) = 2$. Hence, $B_Z \subset (B_{X^*})'_2$, contradiction with (*1). □

Theorem: Let K be a countable compact space with accumulation points K' and put $X = C(K)$. Then $(B_{X^*})'_2 = B_{M(K')}$.

Hence, if (*1) holds and moreover $X = \mathcal{C}(K)$ for some compact space K, then $X \equiv c$.

Characterization among $C(K)$ spaces

Recall:

$$(B_{X^*})'_2 := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq 2\}.$$

If $X \equiv c$, then

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

Assume X satisfies (*1). Then $X^* \equiv \ell_1$.

Proof.

Known results

- If X is separable L_1-predual, then $X^* \not\equiv \ell_1$ iff $L_1[0,1] \subset X^*$ isometrically
- $L_1[0,1]$ has D2P (every weakly open subset of $B_{L_1[0,1]}$ has diameter two)

Assume $Z \subset X^*$ is isometric to $L_1[0,1]$. Pick U be a w^*-open subset of X^* such that $U \cap B_Z \neq \emptyset$. Then $U \cap Z$ is weakly open in Z, so $\operatorname{diam}(U \cap B_Z) = 2$. Hence, $B_Z \subset (B_{X^*})'_2$, contradiction with (*1). □

Theorem: Let K be a countable compact space with accumulation points K' and put $X = C(K)$. Then $(B_{X^*})'_2 = B_{M(K')}$.

Hence, if (*1) holds and moreover $X = \mathcal{C}(K)$ for some compact space K, then $X \equiv c$.

Recall:
$$(B_{X^*})'_2 := \{x^* \in B_{X^*} : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap B_{X^*}) \geq 2\}.$$
If $X \equiv c$, then
$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$
Assume X satisfies (*1). Then $X^* \equiv \ell_1$.

Proof.

Known results

- If X is separable L_1-predual, then $X^* \not\equiv \ell_1$ iff $L_1[0,1] \subset X^*$ isometrically
- $L_1[0,1]$ has D2P (every weakly open subset of $B_{L_1[0,1]}$ has diameter two)

Assume $Z \subset X^*$ is isometric to $L_1[0,1]$. Pick U be a w^*-open subset of X^* such that $U \cap B_Z \neq \varnothing$. Then $U \cap Z$ is weakly open in Z, so $\operatorname{diam}(U \cap B_Z) = 2$. Hence, $B_Z \subset (B_{X^*})'_2$, contradiction with (*1). □

Theorem: Let K be a countable compact space with accumulation points K' and put $X = C(K)$. Then $(B_{X^*})'_2 = B_{M(K')}$.

Hence, if (*1) holds and moreover $X = \mathcal{C}(K)$ for some compact space K, then $X \equiv c$.

Proposition

If $X^* \equiv \ell_1$, then $X \equiv \mathcal{C}(K)$ iff $\{\pm e_n^* \colon n \in \mathbb{N}\} \subset \ell_1 = X^*$ is w^*-closed.

Proof.

"$\Rightarrow$" Easier, because then $X^* \equiv \ell_1(K)$ and so $\{\pm e_n^* \colon n \in \mathbb{N}\} \approx K \cup (-K)$.

"$\Leftarrow$" Put $K_0 := \{\pm e_n^* \colon n \in \mathbb{N}\}$.

By the Baire category theorem (applied to open dense sets $X \setminus \operatorname{Ker} e_n^*$), there is $z \in X$ such that $e_n^*(z) \neq 0$ for every n.

Put

$$K := \{\sigma_n e_n^*;\ n \in \mathbb{N}\}, \quad \text{where } \sigma_n := \operatorname{sgn}(e_n^*(z)).$$

The set K is w^* compact (because $K = \{x^* \in K_0;\ x^*(z) \geq 0\}$).

Consider the mapping $T : X \to C(K)$ given as $Tx(\sigma_n e_n^*) := \sigma_n e_n^*(x)$. This is isometric embedding.

Moreover, $T^* \delta_{\sigma_n e_n^*} = \sigma_n e_n^*$, so T^* is isometry, because finite linear combinations of dirac masses are norm-dense in $M(K)$ and

$$\|T^*(\sum_{n \in F} a_n \delta_{\sigma_n e_n^*})\|_1 = \|\sum_{n \in F} a_n \sigma_n e_n^*\|_1 = \sum_{n \in F} |a_n| = \|\sum_{n \in F} a_n \delta_{\sigma_n e_n^*}\|_{M(K)}, \quad F \subset \mathbb{N} \text{ finite}.$$

Hence, T^* is one-to-one and T is surjective. $\square$

When is ℓ_1 predual isometric to a $C(K)$ space

Proposition

If $X^* \equiv \ell_1$, then $X \equiv \mathcal{C}(K)$ iff $\{\pm e_n^* \colon n \in \mathbb{N}\} \subset \ell_1 = X^*$ is w^*-closed.

Proof.

"$\Rightarrow$" Easier, because then $X^* \equiv \ell_1(K)$ and so $\{\pm e_n^* \colon n \in \mathbb{N}\} \approx K \cup (-K)$.

"$\Leftarrow$" Put $K_0 := \{\pm e_n^* \colon n \in \mathbb{N}\}$.

By the Baire category theorem (applied to open dense sets $X \setminus \operatorname{Ker} e_n^*$), there is $z \in X$ such that $e_n^*(z) \neq 0$ for every n.

Put

$$K := \{\sigma_n e_n^*;\ n \in \mathbb{N}\}, \quad \text{where } \sigma_n := \operatorname{sgn}(e_n^*(z)).$$

The set K is w^* compact (because $K = \{x^* \in K_0;\ x^*(z) \geq 0\}$).

Consider the mapping $T : X \to C(K)$ given as $Tx(\sigma_n e_n^*) := \sigma_n e_n^*(x)$. This is isometric embedding.

Moreover, $T^* \delta_{\sigma_n e_n^*} = \sigma_n e_n^*$, so T^* is isometry, because finite linear combinations of dirac masses are norm-dense in $M(K)$ and

$$\|T^*(\sum_{n \in F} a_n \delta_{\sigma_n e_n^*})\|_1 = \|\sum_{n \in F} a_n \sigma_n e_n^*\|_1 = \sum_{n \in F} |a_n| = \|\sum_{n \in F} a_n \delta_{\sigma_n e_n^*}\|_{M(K)}, \quad F \subset \mathbb{N} \text{ finite}.$$

Hence, T^* is one-to-one and T is surjective. $\square$

Proposition

If $X^* \equiv \ell_1$, then $X \equiv \mathcal{C}(K)$ iff $\{\pm e_n^* \colon n \in \mathbb{N}\} \subset \ell_1 = X^*$ is w^*-closed.

Proof.

"$\Rightarrow$" Easier, because then $X^* \equiv \ell_1(K)$ and so $\{\pm e_n^* \colon n \in \mathbb{N}\} \approx K \cup (-K)$.

"$\Leftarrow$" Put $K_0 := \{\pm e_n^* \colon n \in \mathbb{N}\}$.

By the Baire category theorem (applied to open dense sets $X \setminus \operatorname{Ker} e_n^*$), there is $z \in X$ such that $e_n^*(z) \neq 0$ for every n.

Put

$$K := \{\sigma_n e_n^*;\ n \in \mathbb{N}\}, \quad \text{where } \sigma_n := \operatorname{sgn}(e_n^*(z)).$$

The set K is w^* compact (because $K = \{x^* \in K_0;\ x^*(z) \geq 0\}$).

Consider the mapping $T : X \to C(K)$ given as $Tx(\sigma_n e_n^*) := \sigma_n e_n^*(x)$. This is isometric embedding.

Moreover, $T^* \delta_{\sigma_n e_n^*} = \sigma_n e_n^*$, so T^* is isometry, because finite linear combinations of dirac masses are norm-dense in $M(K)$ and

$$\|T^*(\sum_{n\in F} a_n \delta_{\sigma_n e_n^*})\|_1 = \|\sum_{n\in F} a_n \sigma_n e_n^*\|_1 = \sum_{n\in F} |a_n| = \|\sum_{n\in F} a_n \delta_{\sigma_n e_n^*}\|_{M(K)}, \quad F \subset \mathbb{N} \text{ finite}.$$

Hence, T^* is one-to-one and T is surjective. $\square$

Proposition

If $X^* \equiv \ell_1$, then $X \equiv \mathcal{C}(K)$ iff $\{\pm e_n^* \colon n \in \mathbb{N}\} \subset \ell_1 = X^*$ is w^*-closed.

Proof.

"$\Rightarrow$" Easier, because then $X^* \equiv \ell_1(K)$ and so $\{\pm e_n^* \colon n \in \mathbb{N}\} \approx K \cup (-K)$.

"$\Leftarrow$" Put $K_0 := \{\pm e_n^* \colon n \in \mathbb{N}\}$.

By the Baire category theorem (applied to open dense sets $X \setminus \operatorname{Ker} e_n^*$), there is $z \in X$ such that $e_n^*(z) \neq 0$ for every n.

Put

$$K := \{\sigma_n e_n^*;\ n \in \mathbb{N}\}, \quad \text{where } \sigma_n := \operatorname{sgn}(e_n^*(z)).$$

The set K is w^* compact (because $K = \{x^* \in K_0;\ x^*(z) \geq 0\}$).

Consider the mapping $T : X \to C(K)$ given as $Tx(\sigma_n e_n^*) := \sigma_n e_n^*(x)$. This is isometric embedding.

Moreover, $T^*\delta_{\sigma_n e_n^*} = \sigma_n e_n^*$, so T^* is isometry, because finite linear combinations of dirac masses are norm-dense in $M(K)$ and

$$\|T^*(\sum_{n\in F} a_n \delta_{\sigma_n e_n^*})\|_1 = \|\sum_{n\in F} a_n \sigma_n e_n^*\|_1 = \sum_{n\in F} |a_n| = \|\sum_{n\in F} a_n \delta_{\sigma_n e_n^*}\|_{M(K)}, \quad F \subset \mathbb{N} \text{ finite}.$$

Hence, T^* is one-to-one and T is surjective. □

When is ℓ_1 predual isometric to a $C(K)$ space

Proposition

If $X^* \equiv \ell_1$, then $X \equiv \mathcal{C}(K)$ iff $\{\pm e_n^*\colon n \in \mathbb{N}\} \subset \ell_1 = X^*$ is w^*-closed.

Proof.

"$\Rightarrow$" Easier, because then $X^* \equiv \ell_1(K)$ and so $\{\pm e_n^*\colon n \in \mathbb{N}\} \approx K \cup (-K)$.

"$\Leftarrow$" Put $K_0 := \{\pm e_n^*\colon n \in \mathbb{N}\}$.

By the Baire category theorem (applied to open dense sets $X \setminus \operatorname{Ker} e_n^*$), there is $z \in X$ such that $e_n^*(z) \neq 0$ for every n.

Put

$$K := \{\sigma_n e_n^*;\ n \in \mathbb{N}\}, \quad \text{where } \sigma_n := \operatorname{sgn}(e_n^*(z)).$$

The set K is w^* compact (because $K = \{x^* \in K_0;\ x^*(z) \geq 0\}$).

Consider the mapping $T : X \to C(K)$ given as $Tx(\sigma_n e_n^*) := \sigma_n e_n^*(x)$. This is isometric embedding.

Moreover, $T^*\delta_{\sigma_n e_n^*} = \sigma_n e_n^*$, so T^* is isometry, because finite linear combinations of dirac masses are norm-dense in $M(K)$ and

$$\|T^*(\sum_{n\in F} a_n\delta_{\sigma_n e_n^*})\|_1 = \|\sum_{n\in F} a_n\sigma_n e_n^*\|_1 = \sum_{n\in F}|a_n| = \|\sum_{n\in F} a_n\delta_{\sigma_n e_n^*}\|_{M(K)}, \quad F \subset \mathbb{N} \text{ finite}.$$

Hence, T^* is one-to-one and T is surjective. $\square$

When is ℓ_1 predual isometric to a $C(K)$ space

Proposition

If $X^* \equiv \ell_1$, then $X \equiv \mathcal{C}(K)$ iff $\{\pm e_n^* \colon n \in \mathbb{N}\} \subset \ell_1 = X^*$ is w^*-closed.

Proof.

"$\Rightarrow$" Easier, because then $X^* \equiv \ell_1(K)$ and so $\{\pm e_n^* \colon n \in \mathbb{N}\} \approx K \cup (-K)$.

"$\Leftarrow$" Put $K_0 := \{\pm e_n^* \colon n \in \mathbb{N}\}$.

By the Baire category theorem (applied to open dense sets $X \setminus \operatorname{Ker} e_n^*$), there is $z \in X$ such that $e_n^*(z) \neq 0$ for every n.

Put

$$K := \{\sigma_n e_n^*;\ n \in \mathbb{N}\}, \quad \text{where } \sigma_n := \operatorname{sgn}(e_n^*(z)).$$

The set K is w^* compact (because $K = \{x^* \in K_0;\ x^*(z) \geq 0\}$).

Consider the mapping $T : X \to C(K)$ given as $Tx(\sigma_n e_n^*) := \sigma_n e_n^*(x)$. This is isometric embedding.

Moreover, $T^*\delta_{\sigma_n e_n^*} = \sigma_n e_n^*$, so T^* is isometry, because finite linear combinations of dirac masses are norm-dense in $M(K)$ and

$$\|T^*(\sum_{n\in F} a_n \delta_{\sigma_n e_n^*})\|_1 = \|\sum_{n\in F} a_n \sigma_n e_n^*\|_1 = \sum_{n\in F} |a_n| = \|\sum_{n\in F} a_n \delta_{\sigma_n e_n^*}\|_{M(K)}, \quad F \subset \mathbb{N} \text{ finite.}$$

Hence, T^* is one-to-one and T is surjective. □

When is ℓ_1 predual isometric to a $C(K)$ space

Proposition

If $X^* \equiv \ell_1$, then $X \equiv \mathcal{C}(K)$ iff $\{\pm e_n^* \colon n \in \mathbb{N}\} \subset \ell_1 = X^*$ is w^*-closed.

Proof.

"$\Rightarrow$" Easier, because then $X^* \equiv \ell_1(K)$ and so $\{\pm e_n^* \colon n \in \mathbb{N}\} \approx K \cup (-K)$.

"$\Leftarrow$" Put $K_0 := \{\pm e_n^* \colon n \in \mathbb{N}\}$.

By the Baire category theorem (applied to open dense sets $X \setminus \operatorname{Ker} e_n^*$), there is $z \in X$ such that $e_n^*(z) \neq 0$ for every n.

Put

$$K := \{\sigma_n e_n^*;\ n \in \mathbb{N}\}, \quad \text{where } \sigma_n := \operatorname{sgn}(e_n^*(z)).$$

The set K is w^* compact (because $K = \{x^* \in K_0;\ x^*(z) \geq 0\}$).

Consider the mapping $T : X \to C(K)$ given as $Tx(\sigma_n e_n^*) := \sigma_n e_n^*(x)$. This is isometric embedding.

Moreover, $T^* \delta_{\sigma_n e_n^*} = \sigma_n e_n^*$, so T^* is isometry, because finite linear combinations of dirac masses are norm-dense in $M(K)$ and

$$\|T^*(\sum_{n\in F} a_n \delta_{\sigma_n e_n^*})\|_1 = \|\sum_{n\in F} a_n \sigma_n e_n^*\|_1 = \sum_{n\in F} |a_n| = \|\sum_{n\in F} a_n \delta_{\sigma_n e_n^*}\|_{M(K)}, \quad F \subset \mathbb{N} \text{ finite}.$$

Hence, T^* is one-to-one and T is surjective. $\square$

Proposition

If $X^* \equiv \ell_1$, then $X \equiv \mathcal{C}(K)$ iff $\{\pm e_n^* \colon n \in \mathbb{N}\} \subset \ell_1 = X^*$ is w^*-closed.

Proof.

"$\Rightarrow$" Easier, because then $X^* \equiv \ell_1(K)$ and so $\{\pm e_n^* \colon n \in \mathbb{N}\} \approx K \cup (-K)$.

"$\Leftarrow$" Put $K_0 := \{\pm e_n^* \colon n \in \mathbb{N}\}$.

By the Baire category theorem (applied to open dense sets $X \setminus \operatorname{Ker} e_n^*$), there is $z \in X$ such that $e_n^*(z) \neq 0$ for every n.

Put

$$K := \{\sigma_n e_n^*;\ n \in \mathbb{N}\}, \quad \text{where } \sigma_n := \operatorname{sgn}(e_n^*(z)).$$

The set K is w^* compact (because $K = \{x^* \in K_0;\ x^*(z) \geq 0\}$).

Consider the mapping $T : X \to C(K)$ given as $Tx(\sigma_n e_n^*) := \sigma_n e_n^*(x)$. This is isometric embedding.

Moreover, $T^*\delta_{\sigma_n e_n^*} = \sigma_n e_n^*$, so T^* is isometry, because finite linear combinations of dirac masses are norm-dense in $M(K)$ and

$$\|T^*(\sum_{n\in F} a_n \delta_{\sigma_n e_n^*})\|_1 = \|\sum_{n\in F} a_n \sigma_n e_n^*\|_1 = \sum_{n\in F} |a_n| = \|\sum_{n\in F} a_n \delta_{\sigma_n e_n^*}\|_{M(K)}, \quad F \subset \mathbb{N} \text{ finite}.$$

Hence, T^* is one-to-one and T is surjective. $\square$

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

Recall: Assume X satisfies (*1). Then $X^* \equiv \ell_1$. Moreover,

$$X \equiv c \quad \Leftrightarrow \quad \{\pm e_n \colon n \in \mathbb{N}\} \subset \ell_1 = X^* \text{ is } w^*\text{-closed.}$$

Observation 1: (*1) $\Rightarrow$ $\operatorname{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset [-1,1]z^*$.

Proof: Assume $e_{n_k} \overset{w^*}{\to} x^*$. Consider $y_k := e_{n_{2k}}$ and $z_k := e_{n_{2k+1}}$.Then $\|y_k\| = \|z_k\|$ and $\|y_k - z_k\| = 2$ for every $k \in \mathbb{N}$. Since $y_k \overset{w^*}{\to} x^*$ and $z_k \overset{w^*}{\to} x^*$, we obtain $x^* \in (B_{X^*})'_2 \subset [-1,1]z^*$.

(For example, if $X = c_0 \oplus_\infty c$, then $0 \in \overline{\{\pm e_n \colon n \in \mathbb{N}\}}^{w^*}$.)

Condition which distinguish between c and $c_0 \oplus_\infty c$:

$$\exists a \in S_X\ \forall y \in B_X: \quad \max\{\|a \pm y\|\} = 1 + \|y\|. \tag{*2}$$

(Because for any real numbers a, b we have $\max\{|a+b|,\ |a-b|\} = |a| + |b|$.)

Observation 2: (*1) & (*2) $\Rightarrow$ $\operatorname{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset \{\pm z^*\}$.

Proof: Assume $e_{n_k} \overset{w^*}{\to} tz^*$ and $|t| < 1$. Then $\exists k_0$ with $|e_{n_k}(a)| < 1$, $k \geq k_0$. One shows that

$$\overline{\operatorname{span}}^{w^*}\{e_i;\ i \notin \{n_k:\ k \geq k_0\}\} \neq X^*.$$

Hence, there is $y \in S_X$ with $y|_{\{e_i;\ i \notin \{n_k \colon k \geq k_0\}\}} \equiv 0$. But then

$$2 = \|a \pm y\| = \sup_i |e_i(a \pm a)| = \max\{\|a\|, \sup_{k \geq k_0} |e_{n_k}(a \pm y)|\} < 2 \ldots \text{contradiction.}$$

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

Recall: Assume X satisfies (*1). Then $X^* \equiv \ell_1$. Moreover,

$$X \equiv c \quad \Leftrightarrow \quad \{\pm e_n \colon n \in \mathbb{N}\} \subset \ell_1 = X^* \text{ is } w^*\text{-closed.}$$

Observation 1: (*1) $\Rightarrow$ $\operatorname{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset [-1,1]z^*$.

Proof: Assume $e_{n_k} \xrightarrow{w^*} x^*$. Consider $y_k := e_{n_{2k}}$ and $z_k := e_{n_{2k+1}}$. Then $\|y_k\| = \|z_k\|$ and $\|y_k - z_k\| = 2$ for every $k \in \mathbb{N}$. Since $y_k \xrightarrow{w^*} x^*$ and $z_k \xrightarrow{w^*} x^*$, we obtain $x^* \in (B_{X^*})'_2 \subset [-1,1]z^*$.

(For example, if $X = c_0 \oplus_\infty c$, then $0 \in \overline{\{\pm e_n \colon n \in \mathbb{N}\}}^{w^*}$.)

Condition which distinguish between c and $c_0 \oplus_\infty c$:

$$\exists a \in S_X\ \forall y \in B_X : \quad \max\{\|a \pm y\|\} = 1 + \|y\|. \tag{*2}$$

(Because for any real numbers a, b we have $\max\{|a+b|,\ |a-b|\} = |a| + |b|$.)

Observation 2: (*1) & (*2) $\Rightarrow$ $\operatorname{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset \{\pm z^*\}$.

Proof: Assume $e_{n_k} \xrightarrow{w^*} tz^*$ and $|t| < 1$. Then $\exists k_0$ with $|e_{n_k}(a)| < 1$, $k \geq k_0$. One shows that

$$\overline{\operatorname{span}}^{w^*}\{e_i;\ i \notin \{n_k :\ k \geq k_0\}\} \neq X^*.$$

Hence, there is $y \in S_X$ with $y|_{\{e_i;\ i \notin \{n_k \colon k \geq k_0\}\}} \equiv 0$. But then

$$2 = \|a \pm y\| = \sup_i |e_i(a \pm a)| = \max\{\|a\|, \sup_{k \geq k_0} |e_{n_k}(a \pm y)|\} < 2 \ldots \text{contradiction.}$$

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

Recall: Assume X satisfies (*1). Then $X^* \equiv \ell_1$. Moreover,

$$X \equiv c \quad \Leftrightarrow \quad \{\pm e_n \colon n \in \mathbb{N}\} \subset \ell_1 = X^* \text{ is } w^*\text{-closed.}$$

Observation 1: (*1) $\Rightarrow$ $\mathrm{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset [-1,1]z^*$.

Proof: Assume $e_{n_k} \xrightarrow{w^*} x^*$. Consider $y_k := e_{n_{2k}}$ and $z_k := e_{n_{2k+1}}$. Then $\|y_k\| = \|z_k\|$ and $\|y_k - z_k\| = 2$ for every $k \in \mathbb{N}$. Since $y_k \xrightarrow{w^*} x^*$ and $z_k \xrightarrow{w^*} x^*$, we obtain $x^* \in (B_{X^*})'_2 \subset [-1,1]z^*$.

(For example, if $X = c_0 \oplus_\infty c$, then $0 \in \overline{\{\pm e_n \colon n \in \mathbb{N}\}}^{w^*}$.)

Condition which distinguish between c and $c_0 \oplus_\infty c$:

$$\exists a \in S_X \ \forall y \in B_X : \quad \max\{\|a \pm y\|\} = 1 + \|y\|. \tag{*2}$$

(Because for any real numbers a, b we have $\max\{|a+b|, \ |a-b|\} = |a| + |b|$.)

Observation 2: (*1) & (*2) $\Rightarrow$ $\mathrm{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset \{\pm z^*\}$.

Proof: Assume $e_{n_k} \xrightarrow{w^*} tz^*$ and $|t| < 1$. Then $\exists k_0$ with $|e_{n_k}(a)| < 1$, $k \geq k_0$. One shows that

$$\overline{\mathrm{span}}^{w^*}\{e_i ; \ i \notin \{n_k : \ k \geq k_0\}\} \neq X^*.$$

Hence, there is $y \in S_X$ with $y|_{\{e_i;\ i \notin \{n_k \colon k \geq k_0\}\}} \equiv 0$. But then

$$2 = \|a \pm y\| = \sup_i |e_i(a \pm a)| = \max\{\|a\|, \sup_{k \geq k_0} |e_{n_k}(a \pm y)|\} < 2 \ldots \text{contradiction.}$$

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \qquad (*1)$$

Recall: Assume X satisfies (*1). Then $X^* \equiv \ell_1$. Moreover,

$$X \equiv c \quad \Leftrightarrow \quad \{\pm e_n \colon n \in \mathbb{N}\} \subset \ell_1 = X^* \text{ is } w^*\text{-closed.}$$

Observation 1: (*1) $\Rightarrow$ $\operatorname{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset [-1,1]z^*$.

Proof: Assume $e_{n_k} \overset{w^*}{\to} x^*$. Consider $y_k := e_{n_{2k}}$ and $z_k := e_{n_{2k+1}}$.Then $\|y_k\| = \|z_k\|$ and $\|y_k - z_k\| = 2$ for every $k \in \mathbb{N}$. Since $y_k \overset{w^*}{\to} x^*$ and $z_k \overset{w^*}{\to} x^*$, we obtain $x^* \in (B_{X^*})'_2 \subset [-1,1]z^*$.

(For example, if $X = c_0 \oplus_\infty c$, then $0 \in \overline{\{\pm e_n \colon n \in \mathbb{N}\}}^{w^*}$.)

Condition which distinguish between c and $c_0 \oplus_\infty c$:

$$\exists a \in S_X \; \forall y \in B_X : \quad \max\{\|a \pm y\|\} = 1 + \|y\|. \qquad (*2)$$

(Because for any real numbers a, b we have $\max\{|a+b|, \ |a-b|\} = |a| + |b|$.)

Observation 2: (*1) & (*2) $\Rightarrow$ $\operatorname{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset \{\pm z^*\}$.

Proof: Assume $e_{n_k} \overset{w^*}{\to} tz^*$ and $|t| < 1$. Then $\exists k_0$ with $|e_{n_k}(a)| < 1$, $k \geq k_0$. One shows that

$$\overline{\operatorname{span}}^{w^*}\{e_i;\ i \notin \{n_k : \ k \geq k_0\}\} \neq X^*.$$

Hence, there is $y \in S_X$ with $y|_{\{e_i;\ i \notin \{n_k \colon k \geq k_0\}\}} \equiv 0$. But then

$$2 = \|a \pm y\| = \sup_i |e_i(a \pm a)| = \max\{\|a\|, \sup_{k \geq k_0} |e_{n_k}(a \pm y)|\} < 2 \ldots \text{contradiction.}$$

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

Recall: Assume X satisfies (*1). Then $X^* \equiv \ell_1$. Moreover,

$$X \equiv c \quad \Leftrightarrow \quad \{\pm e_n \colon n \in \mathbb{N}\} \subset \ell_1 = X^* \text{ is } w^*\text{-closed.}$$

Observation 1: (*1) $\Rightarrow$ $\operatorname{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset [-1,1]z^*$.

Proof: Assume $e_{n_k} \overset{w^*}{\to} x^*$. Consider $y_k := e_{n_{2k}}$ and $z_k := e_{n_{2k+1}}$.Then $\|y_k\| = \|z_k\|$ and $\|y_k - z_k\| = 2$ for every $k \in \mathbb{N}$. Since $y_k \overset{w^*}{\to} x^*$ and $z_k \overset{w^*}{\to} x^*$, we obtain $x^* \in (B_{X^*})'_2 \subset [-1,1]z^*$.

(For example, if $X = c_0 \oplus_\infty c$, then $0 \in \overline{\{\pm e_n \colon n \in \mathbb{N}\}}^{w^*}$.)

Condition which distinguish between c and $c_0 \oplus_\infty c$:

$$\exists a \in S_X\ \forall y \in B_X: \quad \max\{\|a \pm y\|\} = 1 + \|y\|. \tag{*2}$$

(Because for any real numbers a, b we have $\max\{|a+b|,\ |a-b|\} = |a| + |b|$.)

Observation 2: (*1) & (*2) $\Rightarrow$ $\operatorname{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset \{\pm z^*\}$.

Proof: Assume $e_{n_k} \overset{w^*}{\to} tz^*$ and $|t| < 1$. Then $\exists k_0$ with $|e_{n_k}(a)| < 1$, $k \geq k_0$. One shows that

$$\overline{\operatorname{span}}^{w^*}\{e_i;\ i \notin \{n_k : k \geq k_0\}\} \neq X^*.$$

Hence, there is $y \in S_X$ with $y|_{\{e_i;\ i \notin \{n_k \colon k \geq k_0\}\}} \equiv 0$. But then

$$2 = \|a \pm y\| = \sup_i |e_i(a \pm a)| = \max\{\|a\|, \sup_{k \geq k_0} |e_{n_k}(a \pm y)|\} < 2 \ldots \text{contradiction.}$$

Spaces we need to distinguish from c

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \qquad (*1)$$

Recall: Assume X satisfies (*1). Then $X^* \equiv \ell_1$. Moreover,

$$X \equiv c \quad \Leftrightarrow \quad \{\pm e_n \colon n \in \mathbb{N}\} \subset \ell_1 = X^* \text{ is } w^*\text{-closed.}$$

Observation 1: (*1) $\Rightarrow$ $\operatorname{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset [-1,1]z^*$.

Proof: Assume $e_{n_k} \overset{w^*}{\to} x^*$. Consider $y_k := e_{n_{2k}}$ and $z_k := e_{n_{2k+1}}$.Then $\|y_k\| = \|z_k\|$ and $\|y_k - z_k\| = 2$ for every $k \in \mathbb{N}$. Since $y_k \overset{w^*}{\to} x^*$ and $z_k \overset{w^*}{\to} x^*$, we obtain $x^* \in (B_{X^*})'_2 \subset [-1,1]z^*$.

(For example, if $X = c_0 \oplus_\infty c$, then $0 \in \overline{\{\pm e_n \colon n \in \mathbb{N}\}}^{w^*}$.)

Condition which distinguish between c and $c_0 \oplus_\infty c$:

$$\exists a \in S_X \; \forall y \in B_X : \quad \max\{\|a \pm y\|\} = 1 + \|y\|. \qquad (*2)$$

(Because for any real numbers a, b we have $\max\{|a+b|, \; |a-b|\} = |a| + |b|$.)

Observation 2: (*1) & (*2) $\Rightarrow$ $\operatorname{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset \{\pm z^*\}$.

Proof: Assume $e_{n_k} \overset{w^*}{\to} tz^*$ and $|t| < 1$. Then $\exists k_0$ with $|e_{n_k}(a)| < 1$, $k \geq k_0$. One shows that

$$\overline{\operatorname{span}}^{w^*}\{e_i; \; i \notin \{n_k : \; k \geq k_0\}\} \neq X^*.$$

Hence, there is $y \in S_X$ with $y|_{\{e_i;\, i \notin \{n_k \colon k \geq k_0\}\}} \equiv 0$. But then

$$2 = \|a \pm y\| = \sup_i |e_i(a \pm a)| = \max\{\|a\|, \sup_{k \geq k_0} |e_{n_k}(a \pm y)|\} < 2 \ldots \text{contradiction.}$$

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \qquad (*1)$$

Recall: Assume X satisfies (*1). Then $X^* \equiv \ell_1$. Moreover,

$$X \equiv c \quad \Leftrightarrow \quad \{\pm e_n \colon n \in \mathbb{N}\} \subset \ell_1 = X^* \text{ is } w^*\text{-closed.}$$

Observation 1: (*1) $\Rightarrow$ $\mathrm{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset [-1,1]z^*$.

Proof: Assume $e_{n_k} \overset{w^*}{\to} x^*$. Consider $y_k := e_{n_{2k}}$ and $z_k := e_{n_{2k+1}}$. Then $\|y_k\| = \|z_k\|$ and $\|y_k - z_k\| = 2$ for every $k \in \mathbb{N}$. Since $y_k \overset{w^*}{\to} x^*$ and $z_k \overset{w^*}{\to} x^*$, we obtain $x^* \in (B_{X^*})'_2 \subset [-1,1]z^*$.

(For example, if $X = c_0 \oplus_\infty c$, then $0 \in \overline{\{\pm e_n \colon n \in \mathbb{N}\}}^{w^*}$.)

Condition which distinguish between c and $c_0 \oplus_\infty c$:

$$\exists a \in S_X\ \forall y \in B_X: \quad \max\{\|a \pm y\|\} = 1 + \|y\|. \qquad (*2)$$

(Because for any real numbers a, b we have $\max\{|a+b|,\ |a-b|\} = |a| + |b|$.)

Observation 2: (*1) & (*2) $\Rightarrow$ $\mathrm{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset \{\pm z^*\}$.

Proof: Assume $e_{n_k} \overset{w^*}{\to} tz^*$ and $|t| < 1$. Then $\exists k_0$ with $|e_{n_k}(a)| < 1$, $k \geq k_0$. One shows that

$$\overline{\mathrm{span}}^{w^*}\{e_i;\ i \notin \{n_k:\ k \geq k_0\}\} \neq X^*.$$

Hence, there is $y \in S_X$ with $y|_{\{e_i;\ i \notin \{n_k \colon k \geq k_0\}\}} \equiv 0$. But then

$$2 = \|a \pm y\| = \sup_i |e_i(a \pm a)| = \max\{\|a\|, \sup_{k \geq k_0} |e_{n_k}(a \pm y)|\} < 2 \ldots \text{contradiction.}$$

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \qquad (*1)$$

Recall: Assume X satisfies (*1). Then $X^* \equiv \ell_1$. Moreover,

$$X \equiv c \quad \Leftrightarrow \quad \{\pm e_n \colon n \in \mathbb{N}\} \subset \ell_1 = X^* \text{ is } w^*\text{-closed}.$$

Observation 1: (*1) $\Rightarrow$ $\mathrm{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset [-1,1]z^*$.

Proof: Assume $e_{n_k} \overset{w^*}{\to} x^*$. Consider $y_k := e_{n_{2k}}$ and $z_k := e_{n_{2k+1}}$.Then $\|y_k\| = \|z_k\|$ and $\|y_k - z_k\| = 2$ for every $k \in \mathbb{N}$. Since $y_k \overset{w^*}{\to} x^*$ and $z_k \overset{w^*}{\to} x^*$, we obtain $x^* \in (B_{X^*})'_2 \subset [-1,1]z^*$.

(For example, if $X = c_0 \oplus_\infty c$, then $0 \in \overline{\{\pm e_n \colon n \in \mathbb{N}\}}^{w^*}$.)

Condition which distinguish between c and $c_0 \oplus_\infty c$:

$$\exists a \in S_X\ \forall y \in B_X: \quad \max\{\|a \pm y\|\} = 1 + \|y\|. \qquad (*2)$$

(Because for any real numbers a, b we have $\max\{|a+b|,\ |a-b|\} = |a| + |b|$.)

Observation 2: (*1) & (*2) $\Rightarrow$ $\mathrm{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset \{\pm z^*\}$.

Proof: Assume $e_{n_k} \overset{w^*}{\to} tz^*$ and $|t| < 1$. Then $\exists k_0$ with $|e_{n_k}(a)| < 1$, $k \geq k_0$. One shows that

$$\overline{\mathrm{span}}^{w^*}\{e_i;\ i \notin \{n_k : k \geq k_0\}\} \neq X^*.$$

Hence, there is $y \in S_X$ with $y|_{\{e_i;\ i \notin \{n_k \colon k \geq k_0\}\}} \equiv 0$. But then

$$2 = \|a \pm y\| = \sup_i |e_i(a \pm a)| = \max\{\|a\|, \sup_{k \geq k_0} |e_{n_k}(a \pm y)|\} < 2 \ldots \text{contradiction}.$$

Spaces we need to distinguish from c

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \qquad (*1)$$

Recall: Assume X satisfies (*1). Then $X^* \equiv \ell_1$. Moreover,

$$X \equiv c \quad \Leftrightarrow \quad \{\pm e_n \colon n \in \mathbb{N}\} \subset \ell_1 = X^* \text{ is } w^*\text{-closed.}$$

Observation 1: (*1) $\Rightarrow$ $\operatorname{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset [-1,1]z^*$.

Proof: Assume $e_{n_k} \xrightarrow{w^*} x^*$. Consider $y_k := e_{n_{2k}}$ and $z_k := e_{n_{2k+1}}$.Then $\|y_k\| = \|z_k\|$ and $\|y_k - z_k\| = 2$ for every $k \in \mathbb{N}$. Since $y_k \xrightarrow{w^*} x^*$ and $z_k \xrightarrow{w^*} x^*$, we obtain $x^* \in (B_{X^*})'_2 \subset [-1,1]z^*$.

(For example, if $X = c_0 \oplus_\infty c$, then $0 \in \overline{\{\pm e_n \colon n \in \mathbb{N}\}}^{w^*}$.)

Condition which distinguish between c and $c_0 \oplus_\infty c$:

$$\exists a \in S_X \; \forall y \in B_X : \quad \max\{\|a \pm y\|\} = 1 + \|y\|. \qquad (*2)$$

(Because for any real numbers a, b we have $\max\{|a+b|, \; |a-b|\} = |a| + |b|$.)

Observation 2: (*1) & (*2) $\Rightarrow$ $\operatorname{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset \{\pm z^*\}$.

Proof: Assume $e_{n_k} \xrightarrow{w^*} tz^*$ and $|t| < 1$. Then $\exists k_0$ with $|e_{n_k}(a)| < 1$, $k \geq k_0$. One shows that

$$\overline{\operatorname{span}}^{w^*}\{e_i; \; i \notin \{n_k : \; k \geq k_0\}\} \neq X^*.$$

Hence, there is $y \in S_X$ with $y|_{\{e_i; \; i \notin \{n_k \colon k \geq k_0\}\}} \equiv 0$. But then

$$2 = \|a \pm y\| = \sup_i |e_i(a \pm a)| = \max\{\|a\|, \sup_{k \geq k_0} |e_{n_k}(a \pm y)|\} < 2 \ldots \text{contradiction.}$$

Spaces we need to distinguish from c

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

Recall: Assume X satisfies (*1). Then $X^* \equiv \ell_1$. Moreover,

$$X \equiv c \quad \Leftrightarrow \quad \{\pm e_n \colon n \in \mathbb{N}\} \subset \ell_1 = X^* \text{ is } w^*\text{-closed.}$$

Observation 1: (*1) $\Rightarrow$ $\operatorname{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset [-1,1]z^*$.

Proof: Assume $e_{n_k} \overset{w^*}{\to} x^*$. Consider $y_k := e_{n_{2k}}$ and $z_k := e_{n_{2k+1}}$.Then $\|y_k\| = \|z_k\|$ and $\|y_k - z_k\| = 2$ for every $k \in \mathbb{N}$. Since $y_k \overset{w^*}{\to} x^*$ and $z_k \overset{w^*}{\to} x^*$, we obtain $x^* \in (B_{X^*})'_2 \subset [-1,1]z^*$.

(For example, if $X = c_0 \oplus_\infty c$, then $0 \in \overline{\{\pm e_n \colon n \in \mathbb{N}\}}^{w^*}$.)

Condition which distinguish between c and $c_0 \oplus_\infty c$:

$$\exists a \in S_X \; \forall y \in B_X : \quad \max\{\|a \pm y\|\} = 1 + \|y\|. \tag{*2}$$

(Because for any real numbers a, b we have $\max\{|a+b|,\ |a-b|\} = |a| + |b|$.)

Observation 2: (*1) & (*2) $\Rightarrow$ $\operatorname{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset \{\pm z^*\}$.

Proof: Assume $e_{n_k} \overset{w^*}{\to} tz^*$ and $|t| < 1$. Then $\exists k_0$ with $|e_{n_k}(a)| < 1$, $k \geq k_0$. One shows that

$$\overline{\operatorname{span}}^{w^*}\{e_i;\ i \notin \{n_k : \ k \geq k_0\}\} \neq X^*.$$

Hence, there is $y \in S_X$ with $y|_{\{e_i;\ i \notin \{n_k \colon k \geq k_0\}} \equiv 0$. But then

$$2 = \|a \pm y\| = \sup_i |e_i(a \pm a)| = \max\{\|a\|, \sup_{k \geq k_0} |e_{n_k}(a \pm y)|\} < 2 \ldots \text{contradiction.}$$

Spaces we need to distinguish from c

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

Recall: Assume X satisfies (*1). Then $X^* \equiv \ell_1$. Moreover,

$$X \equiv c \quad \Leftrightarrow \quad \{\pm e_n \colon n \in \mathbb{N}\} \subset \ell_1 = X^* \text{ is } w^*\text{-closed.}$$

Observation 1: (*1) $\Rightarrow$ $\operatorname{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset [-1,1]z^*$.

Proof: Assume $e_{n_k} \xrightarrow{w^*} x^*$. Consider $y_k := e_{n_{2k}}$ and $z_k := e_{n_{2k+1}}$.Then $\|y_k\| = \|z_k\|$ and $\|y_k - z_k\| = 2$ for every $k \in \mathbb{N}$. Since $y_k \xrightarrow{w^*} x^*$ and $z_k \xrightarrow{w^*} x^*$, we obtain $x^* \in (B_{X^*})'_2 \subset [-1,1]z^*$.

(For example, if $X = c_0 \oplus_\infty c$, then $0 \in \overline{\{\pm e_n \colon n \in \mathbb{N}\}}^{w^*}$.)

Condition which distinguish between c and $c_0 \oplus_\infty c$:

$$\exists a \in S_X \; \forall y \in B_X : \quad \max\{\|a \pm y\|\} = 1 + \|y\|. \tag{*2}$$

(Because for any real numbers a, b we have $\max\{|a+b|,\ |a-b|\} = |a| + |b|$.)

Observation 2: (*1) & (*2) $\Rightarrow$ $\operatorname{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \subset \{\pm z^*\}$.

Proof: Assume $e_{n_k} \xrightarrow{w^*} tz^*$ and $|t| < 1$. Then $\exists k_0$ with $|e_{n_k}(a)| < 1$, $k \geq k_0$. One shows that

$$\overline{\operatorname{span}}^{w^*}\{e_i;\ i \notin \{n_k :\ k \geq k_0\}\} \neq X^*.$$

Hence, there is $y \in S_X$ with $y|_{\{e_i;\ i \notin \{n_k \colon k \geq k_0\}\}} \equiv 0$. But then

$$2 = \|a \pm y\| = \sup_i |e_i(a \pm a)| = \max\{\|a\|, \sup_{k \geq k_0} |e_{n_k}(a \pm y)|\} < 2 \ldots \text{contradiction.}$$

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \qquad (*1)$$

Recall: Assume X satisfies (*1). Then $X^* \equiv \ell_1$. Moreover,

$$X \equiv c \quad \Leftrightarrow \quad \{\pm e_n \colon n \in \mathbb{N}\} \subset \ell_1 = X^* \text{ is } w^*\text{-closed}.$$

Observation 1: $(*1) \quad \Rightarrow \quad \operatorname{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \quad \subset \quad [-1,1]z^*$.

Proof: Assume $e_{n_k} \overset{w^*}{\to} x^*$. Consider $y_k := e_{n_{2k}}$ and $z_k := e_{n_{2k+1}}$.Then $\|y_k\| = \|z_k\|$ and $\|y_k - z_k\| = 2$ for every $k \in \mathbb{N}$. Since $y_k \overset{w^*}{\to} x^*$ and $z_k \overset{w^*}{\to} x^*$, we obtain $x^* \in (B_{X^*})'_2 \subset [-1,1]z^*$.

(For example, if $X = c_0 \oplus_\infty c$, then $0 \in \overline{\{\pm e_n \colon n \in \mathbb{N}\}}^{w^*}$.)

Condition which distinguish between c and $c_0 \oplus_\infty c$:

$$\exists a \in S_X\ \forall y \in B_X: \quad \max\{\|a \pm y\|\} = 1 + \|y\|. \qquad (*2)$$

(Because for any real numbers a, b we have $\max\{|a+b|,\ |a-b|\} = |a| + |b|$.)

Observation 2: (*1) & (*2) $\quad \Rightarrow \quad \operatorname{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \quad \subset \quad \{\pm z^*\}$.

Proof: Assume $e_{n_k} \overset{w^*}{\to} tz^*$ and $|t| < 1$. Then $\exists k_0$ with $|e_{n_k}(a)| < 1$, $k \geq k_0$. One shows that

$$\overline{\operatorname{span}}^{w^*}\{e_i;\ i \notin \{n_k : \ k \geq k_0\}\} \neq X^*.$$

Hence, there is $y \in S_X$ with $y|_{\{e_i;\ i \notin \{n_k \colon k \geq k_0\}} \equiv 0$. But then

$$2 = \|a \pm y\| = \sup_i |e_i(a \pm a)| = \max\{\|a\|, \sup_{k \geq k_0} |e_{n_k}(a \pm y)|\} < 2 \ldots \text{contradiction}.$$

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

$$\exists a \in S_X \ \forall y \in B_X : \quad \max\{\|a \pm y\|\} = 1 + \|y\|. \tag{*2}$$

Recall: Assume X satisfies (*1). Then $X^* \equiv \ell_1$ and $\mathrm{clust}_{w^*}\{\pm e_n \colon n \in \mathbb{N}\} \quad \subset \quad \{\pm z^*\}$.

Moreover,

$$X \equiv c \quad \Leftrightarrow \quad \{\pm e_n \colon n \in \mathbb{N}\} \subset \ell_1 = X^* \text{ is } w^*\text{-closed}.$$

Casini, Miglierina, Piasecki 22': $X^* \equiv \ell_1$ and $e_n \stackrel{w^*}{\to} z \in S_{\ell_1}$, then X is isometric to

$$W_z := \{x \in c \colon \lim_{i\to\infty} x(i) = \sum_{i=1}^{\infty} z(i)x(i)\} \subset c.$$

We arrived to the following ...

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

$$\exists a \in S_X\ \forall y \in B_X: \quad \max\{\|a \pm y\|\} = 1 + \|y\|. \tag{*2}$$

Recall: Assume X satisfies (*1). Then $X^* \equiv \ell_1$ and $\operatorname{clust}_{w^*}\{\pm e_n\colon n \in \mathbb{N}\} \quad \subset \quad \{\pm z^*\}$.

Moreover,

$$X \equiv c \quad \Leftrightarrow \quad \{\pm e_n\colon n \in \mathbb{N}\} \subset \ell_1 = X^* \text{ is } w^*\text{-closed}.$$

Casini, Miglierina, Piasecki 22': $X^* \equiv \ell_1$ and $e_n \overset{w^*}{\to} z \in S_{\ell_1}$, then X is isometric to

$$W_z := \{x \in c\colon \lim_{i\to\infty} x(i) = \sum_{i=1}^{\infty} z(i)x(i)\} \subset c.$$

We arrived to the following ...

$$X \text{ is } L_1\text{-predual and } (B_{X^*})'_2 \subset [-1,1]z^* \text{ for some } z^* \in S_{X^*}. \tag{*1}$$

$$\exists a \in S_X\ \forall y \in B_X: \quad \max\{\|a \pm y\|\} = 1 + \|y\|. \tag{*2}$$

Recall: Assume X satisfies (*1). Then $X^* \equiv \ell_1$ and $\operatorname{clust}_{w^*}\{\pm e_n\colon n \in \mathbb{N}\} \quad \subset \quad \{\pm z^*\}$.

Moreover,

$$X \equiv c \quad \Leftrightarrow \quad \{\pm e_n\colon n \in \mathbb{N}\} \subset \ell_1 = X^* \text{ is } w^*\text{-closed.}$$

Casini, Miglierina, Piasecki 22': $X^* \equiv \ell_1$ and $e_n \stackrel{w^*}{\to} z \in S_{\ell_1}$, then X is isometric to

$$W_z := \{x \in c\colon \lim_{i\to\infty} x(i) = \sum_{i=1}^{\infty} z(i)x(i)\} \subset c.$$

We arrived to the following ...

New condition

Theorem (Main result 1)

Let X be a Banach space such that X^* is isometric to $L_1(\mu)$ for some measure μ. Then there exists a zero-dimensional compact space K such that X is isometric to $C(K)$ if and only if the following condition holds

$$\overline{\mathrm{aco}}\Big\{a \in S_X : \ \max\{\|a-y\|, \|a+y\|\} = 2 \text{ for every } y \in S_X\Big\} = B_X. \tag{1}$$

(Not even separability is assumed. In $C(K)$ spaces the points $a \in S_X$ above are functions attaining values ± 1.)

New condition

Theorem (Main result 1)

Let X be a Banach space such that X^* is isometric to $L_1(\mu)$ for some measure μ. Then there exists a zero-dimensional compact space K such that X is isometric to $C(K)$ if and only if the following condition holds

$$\overline{\text{aco}}\Big\{a \in S_X : \ \max\{\|a-y\|, \|a+y\|\} = 2 \text{ for every } y \in S_X\Big\} = B_X. \tag{1}$$

(Not even separability is assumed. In $C(K)$ spaces the points $a \in S_X$ above are functions attaining values ± 1.)

This implies the following characterization of the Banach space c.

Theorem

Let X be separable Banach space. Then $X \equiv c$ iff

- *X^* is isometric to $L_1(\mu)$,*
- *$(B_{X^*})'_2$ is subset of a 1-dimensional space,*
- $$\forall \varepsilon > 0 : \ \overline{\text{aco}}\Big\{a \in S_X : \ \forall y \in S_X : \ \max\{\|a-y\|, \|a+y\|\} \geq 2-\varepsilon\Big\} = B_X.$$

New condition

Theorem (Main result 1)

Let X be a Banach space such that X^* is isometric to $L_1(\mu)$ for some measure μ. Then there exists a zero-dimensional compact space K such that X is isometric to $C(K)$ if and only if the following condition holds

$$\overline{\mathrm{aco}}\Big\{a \in S_X : \ \max\{\|a-y\|, \|a+y\|\} = 2 \text{ for every } y \in S_X\Big\} = B_X. \tag{1}$$

(Not even separability is assumed. In $C(K)$ spaces the points $a \in S_X$ above are functions attaining values ± 1.)

This implies the following characterization of the Banach space c.

Theorem

Let X be separable Banach space. Then $X \equiv c$ iff

- *X^* is isometric to $L_1(\mu)$,*
- *$(B_{X^*})'_2$ is subset of a 1-dimensional space,*
- $$\forall \varepsilon > 0 : \ \overline{\mathrm{aco}}\Big\{a \in S_X : \ \forall y \in S_X : \ \max\{\|a-y\|, \|a+y\|\} \geq 2-\varepsilon\Big\} = B_X.$$

Corollary

$\langle c \rangle_{\equiv}$ *is* $F_{\sigma,\delta}$.