Structural theorem for polyhedral normed spaces

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Introduction

X real normed space.

- $\mathcal{D}(x) = \{f \in S_{X^*} : f(x) = 1\},\ x \in S_X$
- $A \subseteq X^*, A'$ denotes the w^*-cluster points of A
- $\operatorname{ext} B_X$ denotes the extreme points of the unit ball

Definition

X is polyhedral if the unit ball of every finite-dimensional subspace of X is a polytope.

Polyhedral normed spaces

Definition

Let X be a normed space and consider the following properties:

(IV) $f(x) < 1$ for every $x \in S_X$ and $f \in (\operatorname{ext} B_X)'$;

(V) $\sup\{f(x) \colon \operatorname{ext} B_{X^*} \setminus \mathcal{D}(x)\} < 1$ for every $x \in S_X$;

(VI) every $x \in S_X$ has a neighbourhood V such that for every $y \in V \cap S_X$, the segment $[x, y]$ lies entirely on S_X;

(K) X is polyhedral.

For $i \in \{\mathrm{IV}, \mathrm{V}, \mathrm{VI}, \mathrm{K}\}$, X is called *(i)-polyhedral* if it satisfies property (i).

Definition

X has property (Δ) if for every $x \in S_X$ the set $\operatorname{ext} \mathcal{D}(x)$ is finite.

Faces of the sphere in polyhedral normed spaces

Definition

We say that $F \subseteq S_X$ is a *true face* of B_X if there exists a hyperplane H such that $F = H \cap B_X$ and $\mathrm{int}_H(F) \neq \emptyset$.

Definition

We say that $F \subseteq S_X$ is an *algebraic true face* of B_X if there exists a hyperplane H such that $F = H \cap B_X$ and $\text{a-int}_H(F) \neq \emptyset$.

Faces of the sphere in polyhedral normed spaces

Fact

Let X be a polyhedral normed space, $x \in S_X$ and $f \in \mathcal{D}(x)$, and write $H = f^{-1}(1)$. We have

- $x \in \text{a-int}_H(H \cap B_X) \leftrightarrow x$ is a Gâteaux smooth point.

If, moreover, X is (VI)-polyhedral (or Banach) then

- $x \in \text{a-int}_H(H \cap B_X) \leftrightarrow x \in \text{int}_H(H \cap B_X) \leftrightarrow x$ is Fréchet smooth.

Structural theorem

Theorem (Fonf, 1981)

Let X be a polyhedral Banach space. Then, S_X ban be covered by true faces. Therefore, the set

$$\mathcal{B}_0 := \{f \in S_{X^*} \colon f^{-1}(1) \cap B_X \quad \text{is a true face}\}$$

is a boundary of B_X, which is called the *minimal boundary*.

Question

Can Fonf's theorem be extended to normed spaces?

Example

$\ell_{1,0}$ is a polyhedral normed space but has no algebraic true face.

Structural theorem for normed spaces

Theorem

Let X be a normed space with property (Δ).

- If X (K)-polyhedral then S_X is covered by algebraic true faces of B_X.
- If X is (VI)-polyhedral then S_X is covered by true faces of B_X.

Locally finite tilings

Definition

Let X be a normed space. A family $\mathcal{T}$ of subsets of X is called a **tiling** if

(i) every $T \in \mathcal{T}$ is closed, convex, proper and has nonempty interior (i.e., is a body),

(ii) $X = \bigcup_{T \in \mathcal{T}} T$,

(iii) $\mathrm{int}(T_1) \cap \mathrm{int}(T_2) = \emptyset$ whenever $T_1, T_2 \in \mathcal{T}$ are distinct.

Definition

We say that a tiling is locally finite if every $x \in X$ has a neighbourhood intersecting finitely many tiles.

Locally finite tilings

Theorem

For every normed space X, the following are equivalent:

(a) X has a (VI)-polyhedral renorming with property (Δ)

(b) X admits a locally finite tiling by bounded bodies.

Corollary

A Banach space admits a locally finite tiling by bounded convex bodies $\Leftrightarrow$ it has a (K)-polyhedral renorming with property (Δ).

Theorem (Fonf, 1990)

A separable Banach space admits a locally finite tiling by bounded bodies $\Leftrightarrow$ it has a (K)-polyhedral renorming.

Examples

Example 1

There exists a (K)-polyhedral renorming X of c_{00} that satisfies (Δ) but S_X is not covered by true faces.

Example 2

$X = \ell_{1,0}$ is (V)-polyhedral but it has no algebraic true face.

Renorming result

Theorem

Let X be a Banach space with a fundamental biorthogonal system $\{e_\alpha, \varphi^*_\alpha\}_{\alpha\in\Gamma}$, and let $Y := \operatorname{span}\{e_\alpha\}_{\alpha\in\Gamma}$. Then, Y admits an equivalent norm $|||\cdot|||$ such that

- $(Y, |||\cdot|||)$ is (V)-polyhedral;
- $S_{(Y,|||\cdot|||)}$ has no algebraic true faces, and
- $B_{(Y,|||\cdot|||)} = \operatorname{conv}\big(\operatorname{ext}(B_{(Y,|||\cdot|||)})\big)$.

Construction

Theorem (Dantas, Hájek, Russo, 2023)

Let X be a Banach space with a fundamental biorthogonal system $\{e_\alpha, \varphi^*_\alpha\}_{\alpha\in\Gamma}$, and let Y be the dense subspace of X given by $Y := \mathrm{span}\{e_\alpha\}_{\alpha\in\Gamma}$. Then, Y admits a polyhedral LFC norm.

Proposition

The renorming given there is (IV)-polyhedral.

Thanks!