Divergence operators and Lipschitz-free spaces

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Free space and vanishing charges

The divergence equation

We want to solve the divergence equation

$$\operatorname{div} v = F$$

in an open set $\Omega \subset \mathbb{R}^m$. If $\Omega = \mathbb{R}^m$ and $1 < p < \infty$ then the linear operator

$$\operatorname{div} : \dot{W}_{1,p}(\mathbb{R}^m; \mathbb{R}^m) \to L_p(\mathbb{R}^m)$$

admits a bounded (equivalently: continuous, or uniformly continuous, or Lipschitz) linear right inverse defined by an explicit integral formula.

The special case $p = m$

If $v : \Omega \to \mathbb{R}^m$ is $W_{1,m}$ then v is VMO but not necessarily continuous nor even locally bounded. In fact, if $u(x) = \chi(x) \cdot x_1 \cdot |\log|x|_2|^t$, $0 < t < \frac{m-1}{m}$ and $\chi \in C_c^\infty(\mathbb{R}^m)$ equals 1 at the origin, then $\triangle u \in L_m(\mathbb{R}^m)$, yet $v = \nabla u \notin C_0(\mathbb{R}^m; \mathbb{R}^m)$; in fact v is not locally bounded near the origin. This example is due to Nirenberg.

This raises the following question: Characterize those distributions F such that the equation $\operatorname{div} v = F$ admits a solution $v \in C_0(\Omega; \mathbb{R}^m)$. In particular, does this hold for $F \in L_m(\Omega)$? If so, can v be chosen linearly and continuously in terms of F? Is there a "closed formula" giving v in terms of F?

Two extreme cases

Notice that $C_0(\Omega;\mathbb{R}^m)$ is an $\mathcal{L}_\infty$ Banach space and not a dual (of a Banach space). We consider the space at the other end of the spectrum: $L_1(\Omega;\mathbb{R}^m)$ is an $\mathcal{L}_1$ Banach space and not a dual (of a Banach space).

Our aim is to describe completely the spaces of distributions $X_L(\Omega)$ and $X_C(\Omega)$, equipped with a structure of Banach space, so that the operators

$$\operatorname{div} : L_1(\Omega;\mathbb{R}^m) \to X_L(\Omega) \quad \text{and} \quad \operatorname{div} : C_0(\Omega;\mathbb{R}^m) \to X_C(\Omega)$$

are continuous and surjective.

The case $m = 1$

Consider the distributional derivative $D : \mathcal{D}(\mathbb{R})^* \to \mathcal{D}(\mathbb{R})^*$. We are looking for $X_L(\mathbb{R})$ (resp. $X_C(\mathbb{R})$) such that:

- $D(L_1(\mathbb{R})) \subset X_L(\mathbb{R})$ (resp. $D(C_0(\mathbb{R})) \subset X_C(\mathbb{R})$)
- $D(L_1(\mathbb{R}))$ is dense in $X_L(\mathbb{R})$ (resp. $D(C_0(\mathbb{R}))$ is dense in $X_C(\mathbb{R})$)
- $D(L_1(\mathbb{R}))$ is closed in $X_L(\mathbb{R})$ (resp. $D(C_0(\mathbb{R}))$ is closed in $X_C(\mathbb{R})$)

Our tools will be:

- Guessing X_L (resp. X_C) and their topology; to this end, <u>localized locally convex topologies</u> will be useful
- Hahn-Banach's theorem
- The closed range theorem

The case $m = 1$

$$\begin{array}{ccccccc} L_\infty(\mathbb{R}) & \xleftarrow{D^*} & X_L(\mathbb{R})^* & \qquad & M(\mathbb{R}) & \xleftarrow{D^*} & X_C(\mathbb{R})^* \\ L_1(\mathbb{R}) & \xrightarrow{D} & X_L(\mathbb{R}) & \qquad & C_0(\mathbb{R}) & \xrightarrow{D} & X_C(\mathbb{R}) \end{array}$$

Since $D^* = -D$ (integration by parts and "canceling" functions at infinity) we are led to guess that

$$X_L(\mathbb{R})^* = \mathrm{Lip}_{\mathrm{cst}}(\mathbb{R}) \quad \text{and} \quad X_C(\mathbb{R})^* = BV_{\mathrm{cst}}(\mathbb{R})$$

and

$$\|u\|_L = \mathrm{Lip}(u) = \|u'\|_\infty \quad \text{and} \quad \|u\|_{SCH} = \|Du\| = \|u'\|_1.$$

It remains to determine a predual of $\mathrm{Lip}_{\mathrm{cst}}(\mathbb{R})[\|\cdot\|_L]$ and a predual of $BV_{\mathrm{cst}}(\mathbb{R})[\|\cdot\|_{SCH}]$.

Lipschitz-free space

Due to Arens-Eells (1956) and Godefroy-Kalton (2003). Let (X, x_0) be a pointed metric space. Then

$$\mathrm{Lip}_0(X) = \mathbb{R}^X \cap \{u : u(x_0) = 0 \text{ and } \mathrm{Lip}\, u < \infty\}$$

equipped with $\|u\|_L = \mathrm{Lip}\, u$ is a Banach space. The map

$$\delta : X \to \mathrm{Lip}_0(X)^* : x \mapsto [\![x]\!]$$

is a (non-linear) isometry, where $[\![x]\!] : \mathrm{Lip}_0(X) \to \mathbb{R} : u \mapsto u(x)$ is the evaluation at x. Define the <u>Lipschitz-free space of X</u> as

$$\mathcal{F}(X) = \overline{\mathrm{span}\, \delta(X)}^{\mathrm{Lip}_0(X)^*}$$

Lipschitz-free space

Theorem

$\mathcal{F}(X)^* \cong \mathrm{Lip}_0(X)$.

$$\mathcal{F}(X) \subset \mathrm{Lip}_0(X)^* \cong \mathcal{F}(X)^{**}$$
$$\mathcal{F}(X)^* \cong \mathrm{Lip}_0(X)$$
$$\mathcal{F}(X)$$

Proof.

If $\alpha \in \mathcal{F}(X)^*$ then

$$\alpha \circ \delta : X \xrightarrow{\delta} \mathcal{F}(X) \xrightarrow{\alpha} \mathbb{R}$$

is a member of $\mathrm{Lip}_0(X)$ and $\|\alpha \circ \delta\|_L = \|\alpha\|_{\mathcal{F}^*}$. $\square$

Vanishing strong charges

Charges were introduced Pfeffer (1991) and DP (1997), strong charges are due to DP-Pfeffer (2008) and vanishing strong charges to DP-Torres (2011). Assume $m \geqslant 2$ and define $1^* = \frac{m}{m-1} > 1$. Then

$$BV_{1^*}(\mathbb{R}^m) = L_{1^*}(\mathbb{R}^m) \cap \{u : \text{each } \partial_j u \text{ is a signed Borel measure}\}$$

equipped with $\|u\|_{BV_{1^*}} = \|Du\|(\mathbb{R}^m)$ is a Banach space. The operator

$$\Lambda : L_m(\mathbb{R}^m) \to BV_{1^*}(\mathbb{R}^m)^*$$

is well-defined, injective, and continuous (isoperimetric inequality, equivalently: Gagliardo-Nirenberg-Sobolev). Define the space <u>vanishing strong charges</u> as

$$SCH_0(\mathbb{R}^m) = \overline{\Lambda[L_m(\mathbb{R}^m)]}^{BV_{1^*}(\mathbb{R}^m)^*}.$$

Vanishing strong charges

Theorem

$SCH_0(\mathbb{R}^m)^* \cong BV_{1*}(\mathbb{R}^m)$.

$$SCH_0(\mathbb{R}^m) \subset BV_{1*}(\mathbb{R}^m)^* \cong SCH_0(\mathbb{R}^m)^{**}$$
$$SCH_0(\mathbb{R}^m)^* \cong BV_{1*}(\mathbb{R}^m)$$
$$SCH_0(\mathbb{R}^m)$$

Proof.

If $\alpha \in SCH_0(\mathbb{R}^m)^*$ then

$$\alpha \circ \Lambda : L_m(\mathbb{R}^m) \xrightarrow{\Lambda} SCH_0(\mathbb{R}^m) \xrightarrow{\alpha} \mathbb{R}$$

corresponds to a member of $BV_{1*}(\mathbb{R}^m)$ and $\|\alpha \circ \Lambda\|_{BV_{1*}} = \|\alpha\|_{SCH_0^*}$. □

Divergence of summable vector fields

Theorem (Cúth-Kalenda-Kaplický 2017, Godefroy-Lerner 2018)

$$\begin{array}{ccc} L_\infty(\mathbb{R}^m;\mathbb{R}^m) & \xleftarrow{-\nabla=\mathrm{div}^*} & \mathrm{Lip}_0(\mathbb{R}^m) \\ L_1(\mathbb{R}^m;\mathbb{R}^m) & \xrightarrow{\mathrm{div}} & \mathcal{F}(\mathbb{R}^m). \end{array}$$

(A) $\mathrm{div}^* = -\nabla$.

(B) $\mathrm{div} : L_1(\mathbb{R}^m;\mathbb{R}^m) \to \mathcal{F}(\mathbb{R}^m)$ is surjective.

(C) $\ker(\mathrm{div}) = L_1(\mathbb{R}^m;\mathbb{R}^m) \cap Z_m(\mathbb{R}^m)$.

(D) $\mathcal{F}(\mathbb{R}^m) \cong \frac{L_1(\mathbb{R}^m;\mathbb{R}^m)}{L_1 \cap Z_m(\mathbb{R}^m)}$.

$Z_m(\mathbb{R}^m)$ consists of those $v \in L_{1,\mathrm{loc}}(\mathbb{R}^m)$ whose distributional divergence vanishes.

Step 1

If $v \in L_1(\mathbb{R}^m; \mathbb{R}^m)$ then

$$\operatorname{div} v : \operatorname{Lip}_0(\mathbb{R}^m) \to \mathbb{R} : u \mapsto -\int_{\mathbb{R}^m} (\nabla u) \bullet v \, d\mathcal{L}^m.$$

We have $\operatorname{div} v \in \operatorname{Lip}_0(\mathbb{R}^m)$ and $\|\operatorname{div} v\|^*_{\operatorname{Lip}_0} \leqslant \|v\|_{L_1}$. If $v \in L_1(\mathbb{R}^m; \mathbb{R}^m)$ and $\varepsilon > 0$ there exists $w \in C_c^\infty(\mathbb{R}^m; \mathbb{R}^m)$ such that $\|v - w\|_{L_1} < \varepsilon$, hence, $\|\operatorname{div} v - \operatorname{div} w\|^*_{\operatorname{Lip}_0} < \varepsilon$. Thus, it suffices to show that $\operatorname{div} w \in \mathcal{F}(\mathbb{R}^m)$. Note that

$$\langle u, \operatorname{div} w \rangle = \int_{\mathbb{R}^m} u \cdot \operatorname{div} w \, d\mathcal{L}^m.$$

In fact, if μ is a signed Borel measure on $\mathbb{R}^m$ such that $\int_{\mathbb{R}^m} |x|_2 \, d|\mu|(x) < \infty$ then $F_\mu : \operatorname{Lip}_0(\mathbb{R}^m) \to \mathbb{R} : u \mapsto \int_{\mathbb{R}^m} u \, d\mu$ is a member of $\mathcal{F}(\mathbb{R}^m)$.

Step 1

Let $\varepsilon > 0$. Choose a compact K such that $\int_{\mathbb{R}^m \setminus K} |x|_2 \, d|\mu|(x) < \varepsilon$. Choose a finite Borel partition $\mathcal{B}$ of K all of whose members have diameter less than ε. For $B \in \mathcal{B}$ set $\theta_B = \mu(B)$ and pick $x_B \in B$ arbitrarily. Then $F = \sum_{B \in \mathcal{B}} \theta_B \cdot [\![x_B]\!] \in \mathcal{F}(\mathbb{R}^m)$ and

$$|\langle u, F_\mu - F \rangle| \leqslant \left| \sum_{B \in \mathcal{B}} \int_B \big(u(x_B) - u(x) \big) \, d\mu(x) \right| + \int_{\mathbb{R}^m \setminus K} |u(x)| \, d|\mu|(x)$$
$$< \varepsilon \cdot \|u\|_L \cdot |\mu|(\mathbb{R}^m) + \varepsilon \cdot \|u\|_L .$$

<u>Conclusion:</u> If $v \in L_1(\mathbb{R}^m; \mathbb{R}^m)$ then $\operatorname{div} v \in \mathcal{F}(\mathbb{R}^m)$.

Step 2

Let $\alpha \in \mathcal{F}(\mathbb{R}^m)^* \cong \mathrm{Lip}_0(\mathbb{R}^m)$ be so that $\langle \mathrm{div}\, v, \alpha \rangle = 0$ for every $v \in L_1(\mathbb{R}^m; \mathbb{R}^m)$. Letting $u \in \mathrm{Lip}_0(\mathbb{R}^m)$ correspond to α, we have

$$0 = \langle \mathrm{div}\, v, \alpha \rangle = \langle u, \mathrm{div}\, v \rangle = -\int_{\mathbb{R}^m} (\nabla u) \bullet v \, d\mathcal{L}^m$$

for all v, thus, $\nabla u = 0$ a.e., hence, $u = 0$ and, in turn, $\alpha = 0$. Applying Hahn-Banach's theorem, we draw the

<u>Conclusion:</u> $\mathrm{div}[L_1(\mathbb{R}^m; \mathbb{R}^m)]$ is dense in $\mathcal{F}(\mathbb{R}^m)$.

Step 3

Theorem (Cúth-Kalenda-Kaplický 2017, Godefroy-Lerner 2018)

$$\begin{array}{ccc} L_\infty(\mathbb{R}^m;\mathbb{R}^m) & \xleftarrow{-\nabla=\mathrm{div}^*} & \mathrm{Lip}_0(\mathbb{R}^m) \\ L_1(\mathbb{R}^m;\mathbb{R}^m) & \xrightarrow{\mathrm{div}} & \mathcal{F}(\mathbb{R}^m). \end{array}$$

(A) $\mathrm{div}^* = -\nabla.$

Proof.

Given $u \in \mathrm{Lip}_0(\mathbb{R}^m)$, $\mathrm{div}^*(u)$ corresponds to some $w \in L_\infty(\mathbb{R}^m;\mathbb{R}^m)$ such that

$$\int_{\mathbb{R}^m} w \cdot v \, d\mathcal{L}^m = \langle v, \mathrm{div}^*(u)\rangle = \langle u, \mathrm{div}\, v\rangle = -\int_{\mathbb{R}^m} (\nabla u) \cdot v \, d\mathcal{L}^m$$

for all $v \in L_1(\mathbb{R}^m;\mathbb{R}^m)$. $\square$

Step 4

Theorem (Cúth-Kalenda-Kaplický 2017, Godefroy-Lerner 2018)

$$L_\infty(\mathbb{R}^m;\mathbb{R}^m) \xleftarrow{-\nabla=\mathrm{div}^*} \mathrm{Lip}_0(\mathbb{R}^m)$$
$$L_1(\mathbb{R}^m;\mathbb{R}^m) \xrightarrow{\mathrm{div}} \mathcal{F}(\mathbb{R}^m).$$

(B) $\mathrm{div} : L_1(\mathbb{R}^m;\mathbb{R}^m) \to \mathcal{F}(\mathbb{R}^m)$ is surjective.

Proof.

div^* is an isometry: $\|u\|_L = \mathrm{Lip}\, u = \|\nabla u\|_{L_\infty}$ for all $u \in \mathrm{Lip}_0(\mathbb{R}^m)$. Thus, the range of div^* is closed and so is the range of div, according to the closed range theorem. $\square$

Divergence of continuous vector fields vanishing at infinity

Theorem (DP-Torres 2011)

$$\begin{array}{ccc} M(\mathbb{R}^m;\mathbb{R}^m) & \xleftarrow{-\nabla=\mathrm{div}^*} & BV_{1*}(\mathbb{R}^m) \\ C_0(\mathbb{R}^m;\mathbb{R}^m) & \xrightarrow{\mathrm{div}} & SCH_0(\mathbb{R}^m). \end{array}$$

(A) $\mathrm{div}^* = -\nabla$.

(B) $\mathrm{div} : C_0(\mathbb{R}^m;\mathbb{R}^m) \to SCH_0(\mathbb{R}^m)$ is surjective.

(C) $\ker(\mathrm{div}) = C_0(\mathbb{R}^m;\mathbb{R}^m) \cap Z_m(\mathbb{R}^m)$.

(D) $SCH_0(\mathbb{R}^m) \cong \frac{C_0(\mathbb{R}^m;\mathbb{R}^m)}{C_0 \cap Z_m(\mathbb{R}^m)}$.

Non invertibility

Theorem (Naor-Schechtman 2007)

The operator div : $L_1(\mathbb{R}^m; \mathbb{R}^m) \to \mathcal{F}(\mathbb{R}^m)$ does not admit a right inverse (even a non-linear uniformly continuous one).

Theorem (Bourgain-Brezis 2003, Bouafia 2011)

The operator div : $C_0(\mathbb{R}^m; \mathbb{R}^m) \to SCH_0(\mathbb{R}^m)$ does not admit a right inverse (even a non-linear uniformly continuous one).

Normal currents and Smirnov's theorem

Differential forms

- $0 \leqslant m \leqslant n$ are integers
- $\bigwedge^m \mathbb{R}^n$ is the space of m-forms in $\mathbb{R}^n$; $\bigwedge^0 \mathbb{R}^n \cong \mathbb{R}$; $\bigwedge^n \mathbb{R}^n \cong \mathbb{R}$; $\bigwedge^m \mathbb{R}^n$ is spanned by
$$dx_{\lambda(1)} \wedge \cdots \wedge dx_{\lambda(m)}$$
where $\lambda \in \Lambda(n, m)$ is the set of increasing functions $\{1, \ldots, m\} \to \{1, \ldots, n\}$; it is an inner product space
- $\mathcal{D}^m_K(\mathbb{R}^n)$ consists of all C^∞ differential forms of degree m
$$\omega : \mathbb{R}^n \to \bigwedge^m \mathbb{R}^n$$
supported in a compact set K; it is equipped with the (Fréchet) topology of uniform convergence of partial derivatives of any order
- $\mathcal{D}^m(\mathbb{R}^n) = \cup_K \mathcal{D}_K(\mathbb{R}^n)$ is equipped with the inductive limit topology

Currents in the sense of de Rham

- $\mathcal{D}_m(\mathbb{R}^n)$ is the dual of $\mathcal{D}^m(\mathbb{R}^n)$; its members are called m-currents or m-dimensional currents
- The support of $T \in \mathcal{D}_m(\mathbb{R}^n)$ is the complement of the largest open set $U \subset \mathbb{R}^n$ such that $\operatorname{spt} \omega \subset U \Rightarrow \langle \omega, T \rangle = 0$
- 0-dimensional currents are distributions

Real polyhedral chains

Let $x_0, x_1, \dots, x_m$ and

$$\xi = \frac{(x_1 - x_0) \wedge \cdots \wedge (x_m - x_0)}{|(x_1 - x_0) \wedge \cdots \wedge (x_m - x_0)|} \in \bigwedge_m \mathbb{R}^n.$$

Define $[\![x_0, x_1, \dots, x_m]\!] \in \mathcal{D}_m(\mathbb{R}^n)$ by the formula

$$\langle \omega, [\![x_0, x_1, \dots, x_m]\!] \rangle = \int_{\mathrm{co}\{x_0, x_1, \dots, x_m\}} \langle \xi, \omega(x) \rangle \, d\mathcal{H}^m(x).$$

We let $\mathcal{P}_m(\mathbb{R}^m)$ be the linear subspace of $\mathcal{D}_m(\mathbb{R}^n)$ generated by $[\![x_0, x_1, \dots, x_m]\!]$ corresponding to all choices of $x_0, x_1, \dots, x_m$ and we call its members the m-dimensional polyhedral chains with real coefficients.

Boundary operator

The exterior derivative $d : \mathcal{D}^m(\mathbb{R}^n) \to \mathcal{D}^{m+1}(\mathbb{R}^n)$ is defined by

$$d\left(\sum_{\lambda\in\Lambda(n,m)} w_\lambda dx_{\lambda(1)} \wedge \cdots \wedge dx_{\lambda(m)}\right) = \sum_{i=1}^{n}\sum_{\lambda\in\Lambda(n,m)} (\partial_i w_\lambda) dx_i \wedge dx_{\lambda(1)} \wedge \cdots \wedge dx_{\lambda(m)}.$$

Its adjoint, the boundary operator $\partial : \mathcal{D}_m(\mathbb{R}^n) \to \mathcal{D}_{m-1}(\mathbb{R}^n)$, $m \geqslant 1$, satisfies

$$\partial[\![x_0, x_1, \dots, x_m]\!] = \sum_{j=0}^{m} (-1)^j [\![x_0, x_1, \dots, \widehat{x_j}, \dots, x_m]\!],$$

by the Stokes-Cartan theorem.

Mass and normal mass of a current

We let the mass of $T \in \mathcal{D}_m(\mathbb{R}^n)$ by

$$\mathsf{M}(T) = \sup\{|\langle \omega, T\rangle| : \omega \in \mathcal{D}^m(\mathbb{R}^n) \text{ and } \|\omega\|_\infty \leqslant 1\} \in [0, \infty].$$

For instance

$$\mathsf{M}([\![x_0, x_1, \dots, x_m]\!]) = \mathcal{H}^m\big(\mathrm{co}\{x_0, x_1, \dots, x_m\}\big).$$

It may occur that $\mathsf{M}(T) < \infty$ yet $\mathsf{M}(\partial T) < \infty$ (think of a snowflake). We define

$$\mathsf{N}(T) = \begin{cases} \mathsf{M}(T) + \mathsf{M}(\partial T) & \text{if } m \geqslant 1 \\ \mathsf{M}(T) & \text{if } m = 0 \end{cases}$$

and we say that T is a normal m-current if $\mathsf{N}(T) < \infty$. The space of normal m-currents is denoted $\mathsf{N}_m(\mathbb{R}^n)$.

Rectifiable currents

If $M \subset \mathbb{R}^n$ is a C^1 oriented m-dimensional submanifold of finite volume with boundary ∂M then the current $[\![M]\!] \in \mathcal{D}_m(\mathbb{R}^n)$ defined by

$$\langle \omega, [\![M]\!] \rangle = \int_M \langle \tau_x M, \omega(x) \rangle \, d\mathcal{H}^m(x)$$

is normal and $\partial [\![M]\!] = [\![\partial M]\!]$ (Stokes-Cartan).

This generalizes to the case when M is m-rectifiable, i.e. contained in a countable union of C^1 m-dimensional submanifolds, and its boundary (in the sense of currents) is rectifiable as well.

In case $m = 1$, this corresponds to M being a Lipschitz curve.

Weighted foliations

If (S, μ) is a (standard Borel) probability space and

$$s \mapsto M_s$$

is measurable in an appropriate sense and each $[\![M_s]\!]$ is an m-dimensional rectifiable current whose boundary is rectifiable as well then

$$T = \int_T [\![M_s]\!] \, d\mu(s) \in \mathbf{N}_m(\mathbb{R}^n)$$

provided that

$$\int_S \mathbf{M}([\![M_s]\!]) d\mu(s) < \infty \text{ and } \int_S \mathbf{M}([\![\partial M_s]\!]) d\mu(s) < \infty.$$

Smirnov's theorem

Let $T \in \mathsf{N}_1(\mathbb{R}^n)$ be such that $\partial T = [\![b]\!] - [\![a]\!]$, $a \neq b$. Then there exists a (standard Borel) probability space (S, μ) and a (measurable) family of Lipschitz curves $s \to \Gamma_s$, each being a curve from a to b, such that

$$T = \int_S [\![\Gamma_s]\!] \, d\mu(s).$$

$\mathcal{F}(\Omega)$ as a quotient of L_1

Connectedness by Λ-pencils

Let $a, b \in \Omega$ be distinct and $\Lambda > 0$. A Λ-pencil of curves in Ω from a to b is a probability measure μ on the space of oriented Lipschitz curves γ from a to b satisfying the following property: If ν is the Borel measure on $\mathbb{R}^m$ defined by

$$\nu(A) = \int \mathcal{H}^1(A \cap \mathrm{im}(\gamma))\, d\mu(\gamma)$$

then

- $\nu \ll \mathcal{L}^m$
- $\mathrm{spt}(\nu) \subset \overline{\Omega}$
- $\nu(\mathbb{R}^m) \leqslant \Lambda \cdot |b - a|_2$

We say that Ω is connected by Λ-pencils if each distinct $a, b \in \Omega$ admit a Λ-pencil of curves in Ω connecting them.

Free space as a quotient of L_1

Theorem (DP submitted)

The operator $\operatorname{div} : L_1(\Omega;\mathbb{R}^m) \to \mathcal{F}(\Omega)$ is surjective if and only if Ω is connected by Λ-pencils of curves for some $\Lambda > 0$. In this case, the smallest such Λ is the inverse of the largest $\lambda > 0$ such that $\lambda \cdot \|u\|_L \leqslant \|\nabla u\|_{L_\infty}$ for all u.

Proof.

If div is surjective then div* has closed range and there is a $\lambda > 0$ as in the statement. Moreover, if $a, b \in \Omega$ are distinct there exists $v \in L_1(\Omega;\mathbb{R}^m)$ such that $\operatorname{div} v = [\![b]\!] - [\![a]\!]$ and $\|v\|_{L_1} \leqslant (1+\varepsilon)\cdot\lambda^{-1}\cdot|b-a|_2$, as $\|\![\![b]\!] - [\![a]\!]\|\!| = |b-a|_2$.
Interpret v as a 1-dimensional normal current and apply Smirnov's representation theorem. □

Free space and flat chains

If $T \in \mathcal{D}_0(\mathbb{R}^m)$ is a distribution we define

$$G(T) = \inf\{M(S) \mid S \in \mathcal{D}_1(\mathbb{R}^m) \text{ and } T = \partial S\} \in [0, +\infty]$$

and we let $\mathcal{P}_{0,\#}(\mathbb{R}^m)$ consist of those $T_{E,\theta} = \sum_{x \in E} \theta(x) \cdot [\![x]\!]$, where $E \subset \mathbb{R}^m$ is a finite set, $\theta : E \to \mathbb{R}$, and $\sum_{x \in E} \theta(x) = 0$. Note that $G(T_{E,\theta}) \leqslant \sum_{x \in E} |\theta(x)| \cdot |x|_2 < \infty$. We let $\mathcal{G}(\mathbb{R}^m)$ be the G-completion of $\mathcal{P}_{0,\#}(\mathbb{R}^m)$.

Theorem (DP)

$\mathcal{G}(\mathbb{R}^m) \cong \mathcal{F}(\mathbb{R})$. In fact, $\Upsilon : \mathcal{F}(\mathbb{R}^m) \to \mathcal{G}(\mathbb{R}^m)$ defined by $\langle \varphi, \Upsilon(F) \rangle = \langle \varphi - \varphi(0), F \rangle$ is an isometric isomorphism.

Examples

If a distribution F is one of the following types then there exists $v \in L_1(\mathbb{R}^m;\mathbb{R}^m)$ such that $\operatorname{div} v = F$ and $\|v\|_{L_1} \leqslant (1+\varepsilon)\cdot |\!|\!| F |\!|\!|$

- $F = [\![b]\!] - [\![a]\!]$, $|\!|\!| F |\!|\!| = |b-a|_2$
- $F = \mu$ is a signed Borel measure on $\mathbb{R}^m$ such that $\mu(\mathbb{R}^m) = 0$ and

$$|\!|\!| F |\!|\!| \leqslant \int_{\mathbb{R}^m} |x|_2 \, d|\mu|(x) < \infty$$

- $F = \sum_{j=1}^{\infty} \theta_j \cdot \left([\![b_j]\!] - [\![a_j]\!]\right)$ and

$$|\!|\!| F |\!|\!| \leqslant \sum_{j=1}^{\infty} |\theta_j| \cdot |b_j - a_j|_2 < \infty$$