Some Banach spaces with an equivalent 2-rotund norm

Topics in Banach Space Theory
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Rotundity and Uniform Rotundity

Definition

(a) X is R if

$$\forall x, y \in B_X, x \neq y \Rightarrow \|x + y\| < 2.$$

(b) X is UR if $\forall \varepsilon > 0$,

$$\forall x, y \in B_X, \|x - y\| > \varepsilon \Rightarrow \|x + y\| < 2(1 - \delta(\varepsilon)),$$

where $\delta(\varepsilon) > 0$.

Theorem (Klee)

Every separable Banach space admits an R renorming.

Theorem (Amir-Lindenstrauss)

Every reflexive Banach space admits an R renorming.

2-rotundity and Weak 2-rotundity

Definition (V. Milman (1971))

(a) X is 2R if $\forall (x_n) \subset B_X$

$$\lim_{m,n\to\infty} \|x_m + x_n\| = 2 \Rightarrow (x_n) \quad \text{converges in NORM}$$

(b) X is W2R if $\forall (x_n) \subset B_X$

$$\lim_{m,n\to\infty} \|x_m + x_n\| = 2 \Rightarrow (x_n) \quad \text{converges WEAKLY}$$

Remark

$$UR \Rightarrow 2R \Rightarrow W2R \Rightarrow R.$$

2R and W2R are asymptotic properties of X (see recent book of F. Baudier and G. Lancien).

$W2R \Rightarrow$ *reflexive* by a theorem of James

Proof.

Let $x^* \in S_{X^*}$. $\forall n \geqslant 1 \ \exists x_n \in B_X$ s.t. $x^*(x_n) \geqslant 1 - \frac{1}{n}$,

$$\|x_n + x_m\| \geqslant x^*(x_n + x_m) \geqslant 2 - \frac{1}{n} - \frac{1}{m}.$$

so

$$\lim_{m,n\to\infty} \|x_m + x_n\| = 2$$

By W2R, $\exists x \in B_x$ s.t. $x_n \to x$ weakly and

$$x^*(x) = \lim_{n\to\infty} x_n^*(x) = 1.$$

Hence x^* attains its norm. So X is reflexive (James). $\square$

The Day norm

Definition

The Day norm is a symmetric renorming of c_0:

$$\|(a_n)\|_{Day} = (\sum_{n=1}^{\infty} 4^{-n}(a_n^*)^2)^{1/2},$$

where (a_n^*) is the decreasing rearrangement of $(|a_n|)$.

$$\frac{1}{2}\|(a_n)\|_\infty \leqslant \|(a_n)\|_{\mathrm{Day}} \leqslant \frac{1}{\sqrt{3}}\|(a_n)\|_\infty.$$

W2R renormings

Theorem (Hájek-Johanis (2004))

EVERY reflexive Banach space admits a W2R renorming.

Theorem (Hájek-Johanis (2004))

Suppose X is reflexive and has a (possibly uncountable) normalized basis $(e_\gamma)_{\gamma\in\Gamma}$ with biorthogonal functionals $(e^*_\gamma)_{\gamma\in\Gamma}$. Then

$$|||x||| = (\|x\|^2 + \|(e^*_\gamma(x))\|^2_{\mathrm{Day}})^{1/2}$$

is a W2R renorming of X.

Remark

Call $(X, ||| \cdot |||)$ the Day renorming of X.

2R renormings

Theorem (Odell-Schlumprecht (1998))

Every SEPARABLE reflexive Banach space admits a 2R renorming.

Question

Does EVERY reflexive Banach space admit a 2R renorming?

Unconditional and symmetric norms

Definition

Let $(e_n)_{n=1}^{\infty}$ be a Schauder basis for X.

- $(e_n)_{n=1}^{\infty}$ is C-unconditional if $\forall (a_i)$ and signs $\pm$

$$\frac{1}{C}\|\sum \pm a_i e_i\| \leqslant \|\sum a_i e_i\| \leqslant C\|\sum \pm a_i e_i\|$$

- $(e_n)_{n=1}^{\infty}$ is C-symmetric if $\forall (a_i)$ and permutations $\sigma : \mathbb{N} \to \mathbb{N}$.

$$\frac{1}{C}\|\sum \pm a_{\sigma(i)} e_i\| \leqslant \|\sum a_i e_i\| \leqslant C\|\sum \pm a_{\sigma(i)} e_i\|$$

- $(e_n)_{n=1}^{\infty}$ is C-subsymmetric if

$$\frac{1}{C}\|\sum \pm a_i e_{n_i}\| \leqslant \|\sum a_i e_i\| \leqslant C\|\sum \pm a_i e_{n_i}\|$$

$\forall (a_i)$ and $n_1 < n_2 < \dots$.

1-unconditional bases

Theorem (Figiel-Johnson (1973))

Suppose X is superreflexive and has an unconditional (respectively, symmetric) basis $(e_n)_{n=1}^{\infty}$. Then X has a UR renorming s.t. $(e_n)_{n=1}^{\infty}$ is 1-unconditional (respectively, 1-symmetric).

Theorem (D-Kutzarova-Motakis (2024))

Suppose X is reflexive and has an unconditional basis. Then X has a 2R renorming s.t $(e_n)_{n=1}^{\infty}$ is 1-unconditional.

Proof.

We adapt the Odell-Schlumprecht proof! $\square$

A Stronger Result: Refinement of a theorem of Szankowski

Theorem (Szankowski (1973))

Suppose X is reflexive and has an unconditional basis. Then X is isomorphic to a complemented subspace of a reflexive space Y with a symmetric basis.

Theorem (D-Kutzarova-Motakis (2024))

Suppose X is reflexive and has an unconditional basis $(e_n)_{n=1}^{\infty}$. $\exists$ a 2R space Y with a 1-symmetric basis s.t. $(e_n)_{n=1}^{\infty}$ is equivalent to a 1-complemented and 1-unconditional sequence in Y.

1-symmetric bases

Question (D-Kutzarova-Motakis (2024))

Suppose X is reflexive and has a symmetric basis $(e_n)_{n=1}^{\infty}$. Does X have a $2R$ renorming s.t. $(e_n)_{n=1}^{\infty}$ is 1-symmetric?

Remark

- A 1-symmetric basis (e_n) for X generates a 1-symmetric basis $(e_\gamma)_{\gamma\in\Gamma}$ for a nonseparable space Y when Γ is uncountable:

$$\|\sum_{i=1}^{n} a_i e_{\gamma_i}\| := \|\sum_{i=1}^{n} a_i e_i\|$$

  $\forall n \geqslant 1, (\gamma_i)_{i=1}^{n} \subset \Gamma, (a_i)_{i=1}^{n} \subset \mathbb{R}$.
- X is 2R $\Leftrightarrow$ Y is 2R,

A Partial Result

Theorem (D-Kutzarova-Motakis (2024))

Supose X is reflexive and has a symmetric basis $(e_n)_{n=1}^{\infty}$ with lower Boyd index $p_X > 1$. Then X has a 2R renorming s.t $(e_n)_{n=1}^{\infty}$ is 1-symmetric.

Theorem (Tzafriri)

Suppose X has a symmetric basis. T.f.a.e.

- $p_X > 1$
- *the unit vector basis of ℓ_1^n is not uniformly equivalent to disjointly supported equidistributed vectors in X.*

Nonseparable Baernstein spaces

Let Γ be an uncountable set. Let $\mathcal{F}$ be a collection of finite subsets of Γ s.t.

- $\mathcal{F}$ contains all singletons;
- $\mathcal{F}$ is hereditary, i.e., if $F \in \mathcal{F}$ and $G \subseteq F$ then $G \in \mathcal{F}$;
- $\mathcal{F}$ is compact, i.e., $\{1_F \colon F \in \mathcal{F}\}$ is a compact subset of $\{0,1\}^\Gamma$ in the product topology.

Define a norm $\|\cdot\|$ on $c_{00}(\Gamma)$:

$$\|\sum a_\gamma e_\gamma\| = \sup\Big(\sum_{i=1}^{n}\big(\sum_{\gamma\in F_i} |a_\gamma|\big)^2\Big)^{1/2},$$

over all $n \geqslant 1$ and all disjoint $F_i \in \mathcal{F}$ $(1 \leqslant i \leqslant n)$. The

Baernstein space $(B(\mathcal{F}), \|\cdot\|)$ is the completion of $c_{00}(\Gamma)$.

$(e_\gamma)_{\gamma\in\Gamma}$ is a 1-unconditional basis.

2R renormings

Fact

$B(\mathcal{F})$ is reflexive.

Question

Does $B(\mathcal{F})$ have a 2R renorming?

Theorem (D-Kutzarova (2025))

The Day renorming of $B(\mathcal{F})^*$ is 2R.

Some transfinitely defined families (simplified definition)

- Let $\Gamma = \{0, 1\}^{\mathbb{N}}$
- $\mathcal{F}_0 = \{\emptyset\} \cup \{\{p\} : p \in \Gamma\}$.
- If $\alpha = \beta^+$ is a successor ordinal, let $\mathcal{F}_\alpha$ be any hereditary family s.t.

$$\mathcal{F}_\beta \subseteq \mathcal{F}_\alpha$$

  and $A \in \mathcal{F}_\alpha \Rightarrow \exists n \in \mathbb{N}$, $A_i \in \mathcal{F}_\beta$ $(1 \leqslant i \leqslant n)$ s.t.

$$A = \cup_{i=1}^{n} A_i$$

  and s.t. all $p \in A$ start with the same string of n 0's and 1's.
- If α is a limit ordinal ....

Theorem (D-Kutzarova (2025))

For every countable ordinal α, $B(\mathcal{F}_\alpha)$ has a 2R renorming.

Remark

Kutzarova-Troyanski (1982) constructed a Y WITHOUT any uniformly rotund or uniformly differentiable in every direction renorming. In fact $Y = X \oplus X^*$ for a certain subspace X of $B(\mathcal{F}_1)$.

Corollary (D-Kutzarova (2026))

Y has a 2R renorming.

Abstract interpolation spaces

For $j \geqslant 0$, define the generalized Schreier norm

$$\|x\|_j = \sup_{F \in \mathcal{F}_j} \sum_{\gamma \in F} |x_\gamma| \qquad (x \in c_{oo}(\Gamma)).$$

Let $(e_j)_{j=0}^{\infty}$ be a normalized unconditional basis for a reflexive Banach space X.

$$\|x\|_Y := \left\| \sum_{j=0}^{\infty} \frac{\|x\|_j}{2^j} e_j \right\|_X \qquad (x \in c_{00}(\Gamma)).$$

The abstract interpolation space Y is the completion of $c_{00}(\Gamma)$ w.r.t. $\| \cdot \|_Y$.

Remark

This definition generalizes a construction of Argyros-Motakis used to construct a nonseparable reflexive Tsirelson-type space,

Theorem (D-Kutzarova, arxiv 2026)

T.f.a.e.

- *Y is reflexive.*
- *Y HAS a 2R renorming.*

Corollary

The nonseparable reflexive Tsirelson-type space of Argyros-Motakis has a 2R renorming.

Theorem

Y is reflexive if the $\mathcal{F}_j$ families increase suficiently rapidly, which includes the case of the maximal families.

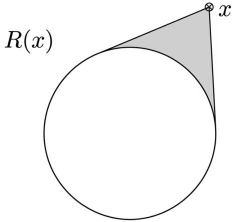
Question

Does Y^* have a 2R renorming?

Property (β) of Rolewicz

For $\|x\| > 1$, let

$$R(x) = \mathrm{conv}(\{x\} \cup B_X) \setminus B_X$$



Theorem (Rolewicz)

$$X \quad \textit{is UR} \quad \Leftrightarrow \quad \mathrm{diam}(R(x)) \to 0 \quad \textit{uniformly as} \quad \|x\| \downarrow 1.$$

Recall the Kuratowski measure of non-compactness of a set $A \subset X$:

$$\alpha(A) = \inf\{\varepsilon > 0 \colon A \subseteq \cup_{i=1}^{n} A_i, \operatorname{diam}(A_i) < \varepsilon\}$$

Definition (Rolewicz (1987))

X has Property (β) if

$$\alpha(R(x)) \to 0 \quad \text{uniformly as} \quad \|x\| \downarrow 1.$$

Theorem (Rolewicz)

$UR \Rightarrow$ Property $(\beta) \Rightarrow$ reflexive.

Examples

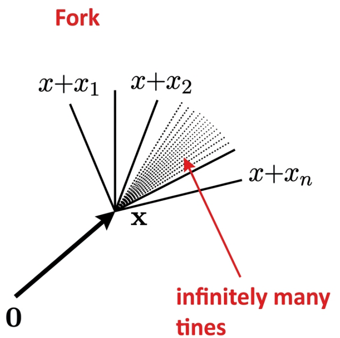
- For $1 < p < \infty$, $(\sum_{n=1}^{\infty} \oplus \ell_{\infty}^{n})_p$ has Property (β).

Theorem (Kutzarova (1990))

TFAE

- *X has Property* (β)
- $\forall \varepsilon > 0 \, \exists \delta > 0$ *s.t.*
  $\forall x \in B_X, (x_n)_{n=1}^{\infty} \subset B_X, \operatorname{sep}(x_n) \geqslant \varepsilon,$

$$\inf_{n \geqslant 1} \frac{\|x + x_n\|}{2} \leqslant 1 - \delta$$



Property (β) and 2R renorming (D-Kutzarova, preliminary)

Theorem

Property (β) $\Rightarrow$ the Day renorming of X has 2R.

Theorem

Suppose X has Property (β) and a subsymmetric basis $(e_n)_{n=1}^{\infty}$. Then X has a 2R renorming s.t. $(e_n)_{n=1}^{\infty}$ is 1-subsymmetric.

Question

Suppose X is reflexive and has a subsymmetric basis $(e_n)_{n=1}^{\infty}$. Does X have a 2R renorming s.t. $(e_n)_{n=1}^{\infty}$ is 1-subsymmetric?

Application: 2R for $\mathcal{L}(\ell_p, \ell_q)$

Theorem (Pitt)

For $1 \leqslant q < p < \infty$, every bounded linear operator from ℓ_p to ℓ_q is compact.

Corollary

For $1 < q < p < \infty$, $\mathcal{L}(\ell_p, \ell_q)$ is reflexive.

Theorem (D-Kutzarova-Randrianarivony-Revalski-Zhivkov (2013))

$\mathcal{L}(\ell_p, \ell_q)$ and its dual have Property (β) $\Leftrightarrow$ $p > 2 > q$.

Corollary

Suppose $p > 2 > q$. The Day renorming of $\mathcal{L}(\ell_p, \ell_q)$ and its dual have 2R.

Theorem

- Let $p > q$. The Day renorming of $\mathcal{L}(\ell_p, \ell_q)^*$ (i.e., $\ell_p \hat{\otimes}_\pi \ell_{q'}$) has 2R.
- Let $p > q \geqslant 2$ or $2 \geqslant p > q$. Then the Day renorming of $\mathcal{L}(\ell_p, \ell_q)$ (i.e., $\ell_p \hat{\otimes}_\varepsilon \ell_{q'}$) does NOT have 2R.

The Day 2R renorming extends to $\mathcal{L}(\ell_p(\Gamma), \ell_q(\Gamma))$, where Γ is an uncountable set.

Remark

If $p > q \geqslant 2$, we do not know if $\mathcal{L}(\ell_p(\Gamma), \ell_q(\Gamma))$ has a 2R renorming.