Lineability of smooth bumps not satisfying Rolle's theorem

Miguel García Bravo
(joint work with Mar Jiménez Sevilla)

Universidad Complutense de Madrid

Summer School 2026: Topics in Banach space Theory
in Castro Urdiales, 09-July-2026

Bump functions

Let X be a Banach space over the field of real numbers.

Definition

A non-zero function $f : X \to \mathbb{R}$ is called a **bump** if f is continuous and

$$\operatorname{supp}_0(f) := \{x \in X : f(x) \neq 0\}$$

is a bounded set. We denote $\operatorname{supp}(f) := \overline{\operatorname{supp}_0(f)})$

1. Every metric space admits Lipschitz bumps.
2. Banach spaces which admit (Fréchet) differentiable bumps.
   - $X = \mathbb{R}^n$.
   - X admits an equivalent differentiable norm $\|\cdot\|$: $c_0(\Gamma)$, $\ell_p(\Gamma)$, $1 < p < \infty$, James space J, ...
   - X is reflexive or X is Asplund and Weakly Compactly Generated.
   - There are Banach spaces with C^1 smooth bumps which do not admit any differentiable equivalent norm (Haydon, 1999).
   - If a Banach space X admits differentiable bumps, then X is Asplund (i.e. all its separable subspaces have separable dual). Due to Ekeland and Lebourg.
3. ℓ_1, ℓ_∞ do not admit differentiable bumps.

Failure of Rolle's theorem on infinite dimensions

Theorem (Rolle, 1691)

For every (continuous) bump $f : \mathbb{R}^n \to \mathbb{R}$ which is differentiable on $U = \text{supp}_0(f)$ there exists a point $x_0 \in U$ such that $f'(x) = 0$.

Theorem (Shkarin, 1992)

Let X be an infinite-dimensional Banach space, which is either

1. *A superreflexive Banach space.*
2. *A non-reflexive Banach space with an equivalent differentiable norm.*

Then, there exists a C^1 bump function $f : X \to \mathbb{R}$ such that $\text{supp}_0(f) = B(0,1)$ and $f'(x) \neq 0$ for all $x \in B(0,1)$.

Failure of Rolle's theorem on infinite dimensions

Example (Shkarin, 1992)

Let $p : L_2[0,1] \to \mathbb{R}$ be defined as

$$p(x) = (1 - \|x\|^2)(\langle Ax, x\rangle + \langle v, x\rangle + 4/27)$$

where $A(x(t)) = tx(t)$ and $v(t) = t(1-t)$.

Example (Ferrer, 1996)

Let $f : \ell_2 \to \mathbb{R}$ be defined as

$$f(x) = \frac{1 - \|x\|^2}{\|x - R(x)\|^2} \quad ; \quad R(x) = \left(\tfrac{1}{2} - \|x\|^2\right) e_1 + (0, x_1, x_2, x_3, \dots).$$

Open question: Does there exist a C^1 function $f : \ell_2 \to \mathbb{R}$ such that $f = 0$ on $\partial B(0,1)$, $f(x) > 0$ for all $\|x\| < 1$ and

$$f'(x) \notin \text{span}(x) \quad \forall x \in B(0,1).$$

Failure of Rolle's theorem on infinite dimensions

Theorem (Azagra, Jiménez-Sevilla 2001)

Let X be an infinite dimensional Banach space. TFAE

1. *X has a C^1 smooth bump.*
2. *There exists a C^1 bump $f : X \to \mathbb{R}$ such that $f'(x) \neq 0$ for all $x \in \mathrm{supp}_0(f)$.*

Remark: Rolle's theorem trivially holds for spaces which do not admit differentiable bumps. And it fails for spaces which have differentiable bumps.

Other 'striking' results in infinite dimensions

Failure of Sard Theorem: There exists a C^∞ function $f : \ell_2(\mathbb{N}) \to \mathbb{R}$ so that $f(C_f) = [0,1]$, where $C_f = \{x \in \ell_2 : \ f'(x) = 0\}$ denotes the set of critical points.

Example ((Kupka, 1969) Bates-Moreira, 2001)

Consider $p : \ell_2(\mathbb{N}) \to \mathbb{R}$ defined as $p(x) = \sum_{n\geq 1}(3 \cdot 2^{-n/3} \cdot x_n^2 - 2 \cdot x_n^3)$.

Failure of Brouwer's Fixed Point Theorem: For the closed unit ball $\overline{B(0,1)} \subset \ell_2(\mathbb{Z})$, there exists a homeomorphism $f : \overline{B(0,1)} \to \overline{B(0,1)}$ with no fixed points.

Example (Kakutani, 1943)

Consider the homeomorphism $f : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ defined as

$$f(x) = \tfrac{1}{2}(1 - \|x\|)e_0 + U(x) \quad ; \quad U(x) = (\ldots, x_{-2}^{-1)}, x_{-1}^{0)}, x_0^{1)}, x_1^{2)}, \ldots)$$

(Bessaga, 1966): There exists a diffeomorphism $h : \ell_2 \to \ell_2 \setminus \{0\}$.

Main question

For those Banach spaces for which Rolle's theorem fails, how many of those smooth bumps without critical points on their open support can we find?

Lineability

Definition

1. A subset $B \subset X$ of a vector space X is called a **Hamel basis** of X if B is formed by linearly independent vectors and every element of X can be written as a finite linear combination of elements of B. Under the Axiom of Choice a Hamel basis of X always exists, and any two Hamel bases of a given vector space have the same cardinality.
2. The **Hamel dimension** of a Banach space X is defined as the cardinality of any of its Hamel basis, which coincides with the cardinality of X.
3. A subset M of a vector space X is said to be τ-**lineable**, for an infinite cardinal τ, if there exists a subspace $N \subset M \cup \{0\}$ of Hamel dimension τ.

History of lineability

The *theory of lineability* studies the existence of large linear structures within nonlinear sets of functions. Some scattered results appeared at the end of the last century, but the term *lineability* was introduced by Gurariy in the early 2000s.

Example (Weierstrass monster function)

Example of a continuous and nowhere differentiable function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \sum_{n=0}^{\infty} \frac{\cos(3^n x)}{2^n}$$

How many examples are of this type?

- (Gurariy, 1966) There exists an infinite dimensional linear space.
- (Fonf, Gurariy, and Kadec, 1999) The above space can be chosen to be closed in $C[0,1]$.

Mathematicians on this topic: Aron, Bastin, Bernal-González,Cariello, Conejero, Dantas, Enflo, Esser, Gurariy Múnoz-Fernández, Pellegrino, Rodríguez-Vidanes, Seoane-Sepúlveda, etc.

Question: If we are given an open subset $U \subset X$ of a Banach space:

1. Can we find a differentiable bump $f : X \to \mathbb{R}$ such that $\text{supp}_0(f) = U$ and $f'(x) \neq 0$?
2. If the answer to (1) is positive. Is such a family of functions lineable?

First we may want to know which Banach spaces admit differentiable bumps with prescribed support U.

- (Wells, 1969) Every separable Banach space with C^1-smooth norm. Proof via smooth partitions of unity.
- (Strömberg, 1996) Every Asplund WCG Banach space. Proof via convolutions.

Up to my knowledge, there seems to still be open questions about understanding zero sets of smooth functions in infinite dimensional Banach spaces.

Results for c_0

Theorem (G-B, 2020)

Let X be an infinite-dimensional separable Banach space with an unconditional basis and which admits an equivalent LFC norm (e.g. c_0). For every open set $U \subset X$, there exists a C^1 bump $f : X \to \mathbb{R}$ such that $\text{supp}_0(f) = U$ and $f'(x) \neq 0$ for all $x \in U$.

Theorem (G-B, Jiménez-Sevilla)

Let X be an infinite-dimensional separable Banach spaces X admitting an equivalent LFC norm (e.g. c_0).
For every open subset $U \subset X$ the following set is $\mathfrak{c}$-lineable

$$\{f \in C^1(X) : \text{supp} f = \overline{U}, \text{ and } f'(x) \neq 0 \text{ for all } x \in U\}$$

Open question: If $U \subset \ell_2$, can we find a C^1 bump $f : \ell_2 \to \mathbb{R}$ so that $\text{supp}_0(f) = U$ and $f'(x) \neq 0$ for all $x \in U$?

Results for non-separable Banach spaces

$$c_0(\Gamma) = \{(x_\gamma)_{\gamma\in\Gamma} \subset \mathbb{R}^\Gamma : \forall\, \varepsilon > 0 \text{ the set } \{\gamma \in \Gamma : |x_\gamma| \geq \varepsilon\} \text{ is finite}\}$$
$$\ell_p(\Gamma) = \{(x_\gamma)_{\gamma\in\Gamma} \subset \mathbb{R}^\Gamma : \sum_{\gamma\in\Gamma} |x_\gamma|^p < \infty\}$$

Theorem (G-B, Jiménez-Sevilla)

Let $X = c_0(\Gamma), \ell_p(\Gamma)$, $1 < p < \infty$, where Γ is a infinite set. Let τ denote the cardinality of a Hamel basis of X^*. Then for every open set $U \subset X$, the following set is τ-lineable:

$$\left\{f \in \mathrm{Diff}(X) \cap C^k(X \setminus \partial U) : \mathit{supp}(f) = \overline{U},\ f'(x) \neq 0\ \forall x \in U\right\}.$$

Here $k \in \mathbb{N}$ denotes the order of smoothness of X.
Obs: *If $\Gamma = \mathbb{N}, \mathbb{R}$ then $\tau = \mathfrak{c}$.*

The proof requires more effort.

1. Consider $g : X \to \mathbb{R}$ C^k bump so that $\text{supp}_0(f) = U$.
2. Find a family $\{f_\alpha\} \subset C^k(U)$ of linearly independent functions so that $|f_\alpha - g| < g/2$ and so that any $f \in \text{span}\{f_\alpha\}$ satisfies $f'(x) \neq 0$ for all $x \in U$.
    - Uses previous results on approximated Morse-Sard type results.
    - For the case $\ell_p(\Gamma)$ we need to use extracting diffeomorphisms.
3. Check that f extended as zero outside U is differentiable on X.

A last result

Definition

1. Let X be a Banach space and $C_{bs}(X)$ denote the family of (continuous) bumps on X. Then, a family $\mathcal{F} \subset C_{bs}(X)$ is said to be **dense** in $C_{bs}(X)$ if for every $h \in C_{bs}(X)$ and every continuous function $\varepsilon : X \to (0, \infty)$, there is $f \in \mathcal{F}$ such that

$$|f(x) - h(x)| \leq \varepsilon(x) \quad \text{for all } x \in X.$$

2. A family $\mathcal{F} \subset C_{bs}(X)$ is called τ-**densely lineable** in $C_{bs}(X)$ if there is a subspace $\mathcal{A} \subset \mathcal{F} \cup \{0\}$ of Hamel dimension τ and dense on $C_{bs}(X)$.

Theorem (G-B, Jiménez-Sevilla)

Let $X = c_0(\Gamma)$. Let τ be the cardinality of $C_{bs}(X)$. Then, the set

$$\{f \in C^\infty(X) : \ f'(x) \neq 0 \ \forall x \in \mathit{supp}_0(f)\}$$

is densely τ-lineable in $C_{bs}(X)$.

Muchas gracias por vuestra atención !!

References:

1. R. M. Aron, L. Bernal González, D. M. Pellegrino, J. B.Seoane Sepúlveda, *Lineability: the search for linearity in mathematics.* Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2016.
2. D. Azagra and M. Jiménez-Sevilla, *The failure of Rolle's theorem in infinite-dimensional Banach spaces*, J. Funct. Anal. 182 (2001), no. 1, 207-226.
3. D. Azagra, M. García-Bravo and M. Jiménez Sevilla, *Approximate Morse-Sard type results for non-separable Banach spaces*, J. Funct. Anal. 287 (2024), no. 4, Paper No. 110488, 50 pp.
4. S. A. Shkarin. *On Rolle's theorem in infinite-dimensional Banach spaces*, Mathematical Notes 51.3 (1992), 311-317.