Norm-attainment in projective tensor products

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Based on joint works with
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Tensor products

Let X, Y, Z vector spaces and denote $B(X \times Y, Z)$ the space of **bilinear** maps $A : X \times Y \to Z$ (if $Z = \mathbb{K}$ we simply write $B(X \times Y)$).

Tensor products

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Consider the linear functional $x \otimes y : B(X \times Y) \to \mathbb{K}$ such that

- $(x \otimes y)(A) = A(x, y), \quad \forall A \in B(X \times Y).$

Definition

The **tensor product** $X \otimes Y$ *is the subspace of the algebraic dual* $B(X \times Y)^{\#}$ *spanned by the set*

$$\{x \otimes y : x \in X,\ y \in Y\}.$$

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A **tensor** $u \in X \otimes Y$ has the following form

$$u = \sum_{i=1}^{n} x_i \otimes y_i, \text{ where } x_i \in X,\ y_i \in Y. \quad \textbf{(NOT UNIQUE)}$$

Linearization $B(X \times Y, Z)$

$$\begin{tikzcd} X \times Y \arrow[r, "A"] \arrow[d, "\otimes"'] & Z \\ X \otimes Y \arrow[ur, dashed, "T_A"'] & \end{tikzcd}$$

Given $A \in B(X \times Y, Z)$, define

$$T_A\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = \sum_{i=1}^{n} A(x_i, y_i).$$

Conversely, if $T : X \otimes Y \to Z$ is linear, define $A \in B(X \times Y, Z)$ by

$$A(x, y) = T(x \otimes y).$$

Projective norm

Definition

Given $X \otimes Y$, we define the **projective norm** *as follows*

$$\|u\|_\pi = \inf\left\{\sum_{i=1}^{n} \|x_i\|\|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i\right\}, \quad \forall u \in X \otimes Y.$$

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However, $(X \otimes Y, \| \cdot \|_\pi)$ is complete if and only if X or Y are finite-dimensional.

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With an analogous linearization diagram as before, it can be proved that

$$\mathcal{B}(X \times Y, Z) = \mathcal{L}(X\widehat{\otimes}_\pi Y, Z)$$

$$\mathcal{B}(X \times Y, \mathbb{K}) = \mathcal{L}(X, Y^*) = (X\widehat{\otimes}_\pi Y)^*.$$

Norm-attainment

Given $u \in X \widehat{\otimes}_\pi Y$, its projective norm admits the following expression

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Definition (S. Dantas, M. Jung, O. Roldán, A. Rueda Zoca, 2022)

Given $u \in X\widehat{\otimes}_\pi Y$ it is said to **attain its projective norm** *if*

$$u = \sum_{i=1}^{\infty} x_i \otimes y_i \quad \text{and} \quad \|u\|_\pi = \sum_{i=1}^{\infty}\|x_i\|\|y_i\|.$$

The set of those u which attain their projective norm is denoted by

$$\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y).$$

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Known results

We have that $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$ in the following cases:

- S. Dantas, M. Jung, O. Roldán, A. Rueda Zoca (2022)
  - If X and Y are finite-dimensional.
  - If $X = \ell_1(I)$ for some set I and Y is any Banach space.
  - If $X = Y$ is a Hilbert space.
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A. Pelczynski, N. Tomczak-Jaegermann (1988)

Given n, m there are spaces X, Y with $\dim(X) = n$ and $\dim(Y) = m$ and $u \in X\widehat{\otimes}_\pi Y$ such that all the optimal representations of u have nm terms (resp. $2nm$ for $\mathbb{K} = \mathbb{C}$).

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There are many examples of X, Y where $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) \neq X\widehat{\otimes}_\pi Y$ (e.g. $L_1(\mathbb{T})\widehat{\otimes}_\pi \ell_2^2$, $L_1[0,1]\widehat{\otimes}_\pi L_1[0,1]$, $\ell_p\widehat{\otimes}_\pi G$ $(1 < p < +\infty)$, $c_0\widehat{\otimes}_\pi \ell_2$).

Lemma (L. C. García-Lirola, J. G.-V., A. Rueda Zoca, 2025)

Let X, Y be Banach spaces such that $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$. Assume $Z \subset Y$ is a 1-complemented subspace. Then, $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Z) = X\widehat{\otimes}_\pi Z$.

This allow us to get rid of the **asumption of duality**, because every dual Banach space Y is 1-complemented in its bidual Y^{**}.

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Theorem (L. C. García-Lirola, J. G.-V., A. Rueda Zoca, 2025)

Let Z be a Banach space such that $Z^* = \ell_1(I)$ isometrically. Let X be a subspace of Z and Y be any Banach space. If either X^* or Y^* has the AP, then $\mathrm{NA}_\pi(X^*\widehat{\otimes}_\pi Y^*) = X^*\widehat{\otimes}_\pi Y^*$.

Key elements of the proof

- The injective tensor product $X\widehat{\otimes}_\varepsilon Y$ is the completion of $X \otimes Y$ endowed with the norm

$$\|u\|_\varepsilon = \sup\left\{\left|\sum_{i=1}^n \varphi(x_i)\psi(y_i)\right| : \ \varphi \in X^*, \ \psi \in Y^*, \ \|\varphi\|, \|\psi\| \leq 1\right\},$$

where $\sum_{i=1}^n x_i \otimes y_i$ is any representation of u.

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Lemma

Let X, Y be normed spaces and $T: X \to Y$ a **linear isometry**. *Then, the operator $T^*: Y^* \to X^*$ is a quotient operator. In fact,* **given** $x^* \in X^*$, **there exists** $y^* \in Y^*$ *such that $T^*(y^*) = x^*$ and $\|y^*\| = \|x^*\|$.*

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Proof.

Take $u \in X^* \widehat{\otimes}_\pi Y^*$.
By the previous lemma, there is $z \in \ell_1(I) \widehat{\otimes}_\pi Y^*$ with

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We can conclude that u attains its norm. □

Consequences: $C(K)$ spaces

Noticing that $C(K)^* = \ell_1(I)$ for K scattered and compact, we obtain:

Corollary (L. C. García-Lirola, J. G.-V., A. Rueda Zoca, 2025)

Let X be a subspace of $C(K)$, where K is a compact Hausdorff scattered space, and Y be any Banach space. If X^* or Y^* has the AP then $\mathrm{NA}_\pi(X^* \widehat{\otimes}_\pi Y^*) = X^* \widehat{\otimes}_\pi Y^*$.

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Example (L. C. García-Lirola, J. G.-V., A. Rueda Zoca, 2025)

The hypothesis of K being **scattered cannot be removed**:
Let $\mathbb{T}$ be the unit circle of $\mathbb{R}^2$. Then,

$$\mathrm{NA}_\pi \left(C(\mathbb{T})^* \widehat{\otimes}_\pi \mathbb{R}^2 \right) \neq C(\mathbb{T})^* \widehat{\otimes}_\pi \mathbb{R}^2.$$

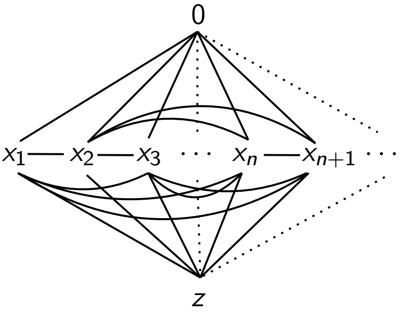
Consequences: Lipschitz-free spaces

Proposition (L. C. García-Lirola, J. G.-V., A. Rueda Zoca, 2025)

Let M be a complete metric space. Then $\mathrm{NA}_\pi(\mathcal{F}(M)\widehat{\otimes}_\pi Y) = \mathcal{F}(M)\widehat{\otimes}_\pi Y$ *for any Banach space Y which is 1-complemented in Y^{**}, if M satisfies one of the following conditions:*

a) *M is countable and proper.*

b) *M is uniformly discrete, countable, and there is a compact Hausdorff topology τ on M such that d is τ-lower semicontinuous, and $V = \{d(x,y) : (x,y) \in M^2\} \subseteq \mathbb{R}_0^+$ is a compact set.*



Example

Define $M = \{0\} \cup \{x_n : n \in \mathbb{N}\} \cup \{z\}$ whose distances are defined as $d(x_n, x_m) = d(x_n, 0) = d(x_n, z) = 1$ for every $n, m \in \mathbb{N}, n \neq m$ and $d(0, z) = 2$.

Define τ such that $(x_n)_n$ converges in τ to x_1 and the points $M \setminus \{x_1\}$ are all discrete.

Then, M satisfies the hypothesis in (b).

Denseness of norm-attaining tensors

Let $f = \sum_{n=1}^{N} \chi_{E_n} \cdot y_n \in L_1(\mu, Y) = L_1(\mu)\widehat{\otimes}_\pi Y$ be a simple function with (E_n) pairwise disjoint sets.

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$$\|f\|_1 = \sum_{n=1}^{N} \mu(E_n)\|y_n\| = \sum_{n=1}^{N} \|\chi_{E_n}\|_1 \cdot \|y_n\|$$

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S. Dantas, M. Jung, O. Roldán, A. Rueda Zoca (2022)

There exist X, Y such that $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ is **NOT** dense in $X\widehat{\otimes}_\pi Y$.

Denseness of norm-attaining tensors

We have that $\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)} = X\widehat{\otimes}_\pi Y$ if:

- S. Dantas, M. Jung, O. Roldán, A. Rueda Zoca (2022)
  - X has the metric π-property, and Y has the metric π-property or Y is uniformly convex.
- S. Dantas, L. C. García-Lirola, M. Jung, A. Rueda Zoca (2023)
  - X and Y are dual spaces with the RNP and at least one of them has the approximation property.
  - X is polyhedral and Y is a dual space.
- L. C. García-Lirola, J. G.-V., A. Rueda Zoca (2025)
  - X has the metric π-property and Y is a dual space with the RNP.
  - $X^* = L_1(\mu)$ and Y is 1-complemented in Y^{**}.

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Optimal representations in L_1-spaces

Recall that $L_1(\mu)\widehat{\otimes}_\pi Y = L_1(\mu, Y)$. Also,

$$\overline{\mathrm{NA}_\pi(L_1(\mu)\widehat{\otimes}_\pi Y)} = L_1(\mu)\widehat{\otimes}_\pi Y \quad \text{and} \quad \mathrm{NA}_\pi(\ell_1(I)\widehat{\otimes}_\pi Y) = \ell_1(I)\widehat{\otimes}_\pi Y.$$

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Is it possible to characterize $\mathrm{NA}_\pi(L_1(\mu)\widehat{\otimes}_\pi Y)$?
When do we have $\mathrm{NA}_\pi(L_1(\mu)\widehat{\otimes}_\pi Y) = L_1(\mu)\widehat{\otimes}_\pi Y$?

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Indeed, this can be formulated without considering tensor products:

- Given $f \in L_1(\mu, Y)$, can we find a representation $f(\omega) = \sum_{n=1}^{\infty} f_n(\omega) y_n$ such that $\|f\| = \sum_{n=1}^{\infty} \|f_i\| \|y_i\|$?

Strictly convex case

Theorem (L. C. García-Lirola, J. G.-V.)

Let (Ω, Σ, μ) *be a measure space and let* Y *be a strictly convex Banach space. Given* $u \in L_1(\mu)\widehat{\otimes}_\pi Y$*, the following assertions are equivalent.*

i) $u \in \mathrm{NA}_\pi(L_1(\mu)\widehat{\otimes}_\pi Y)$.

ii) $u = \sum_{i=1}^{\infty} f_i \otimes y_i$ *with* $\mu\left(\operatorname{supp} f_i \cap \operatorname{supp} f_j\right) = 0$*, for all* $i \neq j$.

iii) *There is a sequence* $(y_n)_{n\in\mathbb{N}} \subseteq S_Y$ *such that* $\mu(\{\omega \in \Omega : u(\omega) \notin \bigcup_{n\in\mathbb{N}} \operatorname{span}\{y_n\}\}) = 0$.

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Corollary (L. C. García-Lirola, J. G.-V.)

Let (Ω, Σ, μ) *be a measure space and let* Y *be a strictly convex Banach space with* $\dim(Y) \geq 2$. *The following statements are equivalent:*

i) $\mathrm{NA}_\pi(L_1(\mu)\widehat{\otimes}_\pi Y) = L_1(\mu)\widehat{\otimes}_\pi Y$.

ii) μ *is purely atomic.*

The case $L_1(\mu)\widehat{\otimes}_\pi L_1(\nu)$

Theorem (L. C. García-Lirola, J. G.-V.)

Let $(\Omega_1, \Sigma_1, \mu)$, and $(\Omega_2, \Sigma_2, \nu)$ be σ-finite measure spaces, and let $f \in L_1(\mu \otimes \nu)$ with $f \geq 0$. Then, the following assertions are equivalent:

i) $f \in \mathrm{NA}_\pi(L_1(\mu)\widehat{\otimes}_\pi L_1(\nu))$.

ii) $f = \sum_{i=1}^{\infty} c_i(\chi_{A_i} \otimes \chi_{B_i})$, *for some* $(c_i)_{i=1}^{\infty} \subseteq [0, \infty)$ *and some measurable rectangles* $(A_i \times B_i)_{i=1}^{\infty} \subseteq \Sigma_1 \times \Sigma_2$.

iii) *For each* $c \geq 0$, *the level set* $\{(\omega, \gamma) \in \Omega_1 \times \Omega_2 : f(\omega, \gamma) > c\}$ *coincides with a countable union of measurable rectangles, up to a* $(\mu \otimes \nu)$*-null set.*

(An analogous result can be stated for any **real-valued** *function).*

The case $L_1(\mu)\widehat{\otimes}_\pi L_1(\nu)$

Theorem (L. C. García-Lirola, J. G.-V.)

Let $(\Omega_1, \Sigma_1, \mu)$, and $(\Omega_2, \Sigma_2, \nu)$ be σ-finite measure spaces, and let $f \in L_1(\mu \otimes \nu)$ with $f \geq 0$. Then, the following assertions are equivalent:

i) $f \in \mathrm{NA}_\pi(L_1(\mu)\widehat{\otimes}_\pi L_1(\nu))$.

ii) $f = \sum_{i=1}^{\infty} c_i(\chi_{A_i} \otimes \chi_{B_i})$, *for some* $(c_i)_{i=1}^{\infty} \subseteq [0, \infty)$ *and some measurable rectangles* $(A_i \times B_i)_{i=1}^{\infty} \subseteq \Sigma_1 \times \Sigma_2$.

iii) *For each* $c \geq 0$, *the level set* $\{(\omega, \gamma) \in \Omega_1 \times \Omega_2 : f(\omega, \gamma) > c\}$ *coincides with a countable union of measurable rectangles, up to a* $(\mu \otimes \nu)$*-null set.*

(An analogous result can be stated for any **real-valued** *function).*

Corollary (L. C. García-Lirola, J. G.-V.)

Let $(\Omega_1, \Sigma_1, \mu)$ and $(\Omega_2, \Sigma_2, \nu)$ be measure spaces. TFAE:

(i) $\mathrm{NA}_\pi(L_1(\mu)\widehat{\otimes}_\pi L_1(\nu)) = L_1(\mu)\widehat{\otimes}_\pi L_1(\nu)$.

(ii) μ *or* ν *is a purely atomic measure.*

Spaces Y such that $\mathrm{NA}_\pi(L_1(\mu)\widehat{\otimes}_\pi Y) = L_1(\mu)\widehat{\otimes}_\pi Y$

Definition (L. C. García-Lirola, J. G.-V.)

Let Y be a real Banach space. We say that Y is **countably convexly generated (CCG)** *if there is a sequence $(e_n)_{n\in\mathbb{N}} \subseteq S_Y$ such that for each $y \in S_Y$ we can find $(\lambda_n)_{n\in\mathbb{N}} \subseteq [0,+\infty)$ with $\sum_{n=1}^\infty \lambda_n = 1$ and $y = \sum_{n=1}^\infty \lambda_n e_n$.*

Theorem (L. C. García-Lirola, J. G.-V.)

Let (Ω, Σ, μ) be a measure space and let Y be a real Banach space. If Y is CCG, then

$$\mathrm{NA}_\pi(L_1(\mu)\widehat{\otimes}_\pi Y) = L_1(\mu)\widehat{\otimes}_\pi Y.$$

Spaces Y such that $\mathrm{NA}_\pi(L_1(\mu)\widehat{\otimes}_\pi Y) = L_1(\mu)\widehat{\otimes}_\pi Y$

Corollary (L. C. García-Lirola, J. G.-V.)

Let (Ω, Σ, μ) be a measure space. If one of the following holds:

- $Y = \mathcal{F}(M)$, *where M is a complete scattered metric space;*
- $Y = C(K)$, *where K is a compact Hausdorff totally disconnected space;*
- $Y = L_\infty(\nu)$, *for any measure ν;*
- $Y = c_0(\Gamma)$, *for any set Γ;*

then $\mathrm{NA}_\pi(L_1(\mu)\widehat{\otimes}_\pi Y) = L_1(\mu)\widehat{\otimes}_\pi Y$.

- L. C. García-Lirola and J. Guerrero-Viu, *Functions in $L_1(\mu, Y)$ with optimal tensor representations* (preprint) https://arxiv.org/abs/2607.02263
- L. C. García-Lirola, J. Guerrero-Viu, and A. Rueda Zoca, *Projective tensor products where every element is norm-attaining*, Banach J. Math. Anal. **19**, No. 19 (2025).

Thank you for your attention!