

Metric Fixed Point Theory

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Introduction to metric fixed point theory

Let (C, d) be a metric space and $T : C \rightarrow C$. T is L -Lipschitz if

$$d(Tx, Ty) \leq Ld(x, y), \quad \forall x, y \in C.$$

$x \in C$ is a fixed point for T if $T(x) = x$.

When can we assure the existence of a fixed point? It depends on the value of L and “metric” or “topological” properties of the domain C .

Theorem (Banach Contraction Principle)

Let (C, d) be a complete m.s. and $T : C \rightarrow C$ L -Lipschitz with $L < 1$ (contraction). Then there exists a unique fixed point.

Hilbert spaces and L -Lipschitz mappings with $L > 1$

For every $L > 1$, there is a L -Lipschitz mapping from the unit ball of a Hilbert space into itself without fixed points.

$$H = \ell_2 = \left\{ x = \{x_n\} \subset \mathbb{R} : \|x\|_2 := \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2} < \infty \right\}$$

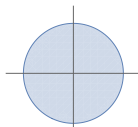
Given $\epsilon > 0$, define $T_\epsilon : B_H \rightarrow B_H$ by

$$T_\epsilon(x_1, x_2, \dots) = (\epsilon(1 - \|x\|_2), x_1, x_2, \dots)$$

Then

$$\|T_\epsilon x - T_\epsilon y\|_2 \leq \sqrt{1 + \epsilon^2} \|x - y\|_2; \quad \forall x, y \in B_H :$$

T_ϵ is $\sqrt{1 + \epsilon^2}$ -Lipschitzian and T_ϵ fails to have a fixed point.



Fixed-point free 1-Lipschitz mappings (nonexpansive)

$T : (\mathbb{R}, |\cdot|) \rightarrow (\mathbb{R}, |\cdot|), T(x) = x + a$ ($a \neq 0$) fixed-point free.

$$c_0 = \{(x_n) \subset \mathbb{R} : \lim_n x_n = 0, \|x\|_\infty = \sup_n |x_n|\}$$

Kakutani Counterexample

$$T : B_{c_0} \rightarrow B_{c_0}; T(x_1, x_2, \dots) = (1, x_1, x_2, \dots)$$

is a fixed-point free isometry $\|Tx - Ty\|_\infty = \|x - y\|_\infty$,
 $x, y \in B_{c_0}$

$$\ell_p := \left\{ x := (t_n) : \|x\|_p = \left[\sum_{n=1}^{\infty} |t_n|^p \right]^{\frac{1}{p}} < +\infty \right\}, \quad 1 \leq p < +\infty$$

Another counter-example

$C = \{x := (t_n) \in \ell_1 : t_n \geq 0, \sum_{n=1}^{\infty} t_n = 1\}$ bounded, closed in ℓ_1 . Let $T : C \rightarrow C$ be given by:

$$T(t_1, t_2, t_3, \dots) = (0, t_1, t_2, t_3, \dots)$$

T is fixed-point free and $\|Tx - Ty\|_1 = \|x - y\|_1$.

If a counterexample can be found in a Banach space X , then it can be lifted to every Banach space Y containing an isometric copy of X .

Approximate fixed point sequences

Let $T : C \rightarrow C$ be nonexpansive, C closed, convex **bounded** of a Banach space X . Then: $\inf\{\|Tx - x\| : x \in C\} = 0$.

A sequence $(x_n) \subset C$ such that $\lim_n \|x_n - Tx_n\| = 0$ is called an **approximate fixed point sequence (afps)**.

Remark: Boundedness hypothesis cannot be dropped

A convex subset $C \subset X$ has the approximate fixed point sequence (AFPP) if for all $T : C \rightarrow C$ nonexpansive, $\inf\{\|x - Tx\| : x \in C\} = 0$.



I. Shafrir, *The approximate fixed point property in Banach spaces and hyperbolic spaces*.

Israel J. of Math. 1990.

A convex subset $C \subset X$ has the AFPP if and only if: for every sequence $(x_n) \subset C$ with $\|x_n\| \rightarrow \infty$ and for every $f \in S_{X^*}$,

$$\limsup_n f\left(\frac{x_n}{\|x_n\|}\right) < 1.$$

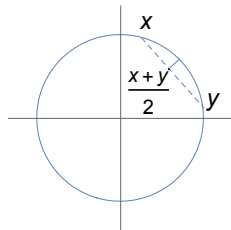
Positive results

X is uniformly convex (UC) if: for very $\epsilon \in (0, 2]$, $\exists \delta = \delta_X(\epsilon) > 0$:

$$\left. \begin{array}{l} \|x\| = 1 \\ \|y\| = 1 \\ \|x - y\| \geq \epsilon \end{array} \right\} \Rightarrow \left\| \frac{x + y}{2} \right\| \leq 1 - \delta_X(\epsilon)$$

Equivalently:

$$\left. \begin{array}{l} \|x_n\| \rightarrow r \\ \|y_n\| \rightarrow r \\ \left\| \frac{x_n + y_n}{2} \right\| \rightarrow r \end{array} \right\} \Rightarrow \|x_n - y_n\| \rightarrow 0.$$



Theorem (F. Browder, 1965)

Let $(X, \|\cdot\|)$ be a Hilbert space or, more generally, a Banach space uniformly convex (UC). Let $C \subset X$ be **closed convex and bounded**. Let $T : C \rightarrow C$ be nonexpansive. Then T has a fixed point.

Proof.

Every UC Banach space is reflexive (closed convex bounded sets are weakly compact).

Let (x_n) be an afps. For all $r > 0$,

$$\{z \in C : \limsup_n \|x_n - z\| \leq r\}$$

is closed convex bounded and T -invariant (and weakly compact)

$r_0 = r(\{x_n\}, C) := \inf\{\limsup_n \|x_n - z\| : z \in C\}$ (Asymp. radius)

$Z_0 = Z(\{x_n\}, C) = \{z \in C : \limsup_n \|x_n - z\| = r_0\}$ (Asymp. center)

$Z_0 = \bigcap_n \{z \in C : \limsup_n \|x_n - z\| \leq r_0 + \frac{1}{n}\}.$

$Z_0 \neq \emptyset$ (by weak-compactness) and T -invariant.

Z_0 is a singleton. Indeed, if $z_1, z_2 \in Z_0$, then

$\limsup_n \|x_n - z_1\| = \limsup_n \|x_n - z_2\| = \limsup_n \left\|x_n - \frac{z_1 + z_2}{2}\right\| = r_0$,
then $\lim_n \|(x_n - z_1) - (x_n - z_2)\| = \|z_1 - z_2\| = 0$ and $z_1 = z_2 = Tz_1$.



Lebesgue spaces

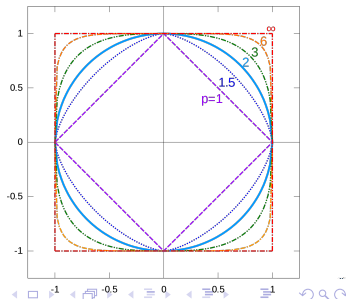
Let (Ω, Σ, μ) a σ -finite measure space, $1 \leq p < +\infty$.

$$L^p(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} : \text{measurable } \|f\|_p = \left[\int_{\Omega} |f(t)|^p d\mu \right]^{1/p} < \infty \right\}$$

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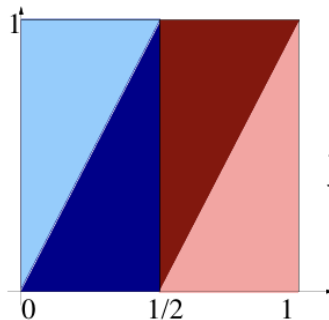
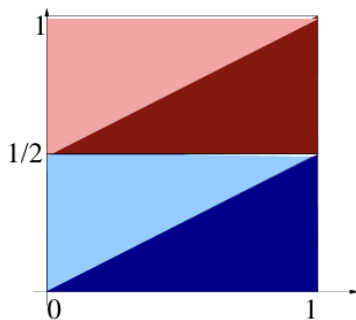
If $1 < p < +\infty$, $L^p(\Omega)$ is UC.

Weak-compactness is not enough.



The baker transform: An example of an ergodic mapping

$$S : [0, 1) \rightarrow [0, 1); S(x, y) = \begin{cases} \left(\frac{x}{2}, 2y\right) & 0 \leq y \leq \frac{1}{2} \\ \left(\frac{x+1}{2}, 2y-1\right) & \frac{1}{2} \leq y < 1 \end{cases}$$



Weak-compactness for the domain is not enough

There exists a convex, weakly compact subset C in $L^1[0, 1]$ and a fixed-point free isometry $T : C \rightarrow C$.

$$C = \left\{ f : [0, 1] \rightarrow [0, 1], \text{ measurable, } \int_0^1 f dm = \frac{1}{2} \right\}. \quad T : C \rightarrow C,$$

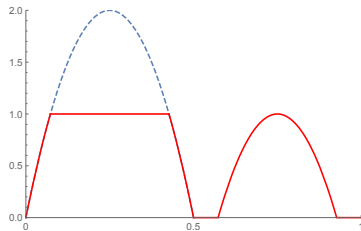
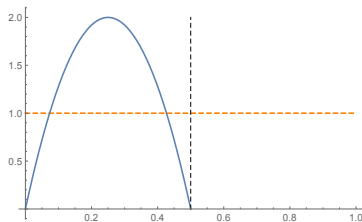
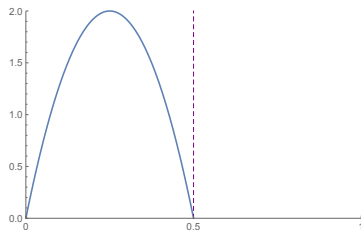
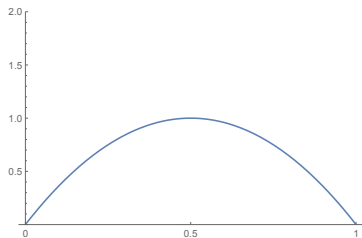
$$Tf(t) = \begin{cases} \min\{2f(2t), 1\} & 0 \leq t \leq \frac{1}{2} \\ \max\{2f(2t-1) - 1, 0\} & \frac{1}{2} < t \leq 1 \end{cases}$$

T is fixed point free in C and $\|Tf - Tg\|_1 = \|f - g\|_1 \quad \forall f, g \in C$.



D. Alspach, *A fixed point free nonexpansive map*, Proc. Amer. Math. Soc., 1981.

Image of $f(t) = -4(t - \frac{1}{2})^2 + 1$ under Alspach's example



Definitions

A subset $C \subset X$ has the FPP if for every $T : C \rightarrow C$ nonexpansive there is a fixed point.

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Problem 1

Does there exist a closed convex [unbounded set](#) with the FPP?

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Definition

A Banach space X is said to have the FPP when every closed convex bounded set has the FPP.

A Banach space X is said to have the w -FPP when every convex weakly compact set has the FPP.

- c_0, ℓ_1 fail the FPP (and every Banach space containing an isometric copy of either c_0 or ℓ_1)
- $L_1[0, 1]$ fails the w -FPP

A super-space for all separable Banach space

$$\ell_\infty := \{x = (x_n) : (x_n) \text{ bounded, } \|x\|_\infty = \sup_n |x_n|\}$$

Every separable B.s. is isometrically embedded in $(\ell_\infty, \|\cdot\|_\infty)$.

Take (x_n) dense in X and $x_n^* \in X^*$ with $x_n^*(x_n) = \|x_n\|$. Then

$X \rightarrow \ell_\infty; \quad x \rightarrow (x_1^*(x), x_2^*(x), \dots, x_n^*(x), \dots)$ isometry, that is

$$\|Tx - Ty\|_\infty = \|x - y\|_X, \quad \forall x, y \in X.$$

ℓ_∞ fails the w -FPP.

Problem 2

Does there exist a Banach space X failing the w -FPP and without isometric copies of $L_1[0, 1]$?

UC $\Rightarrow$ FPP

UC implies reflexivity.

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Problem 3:

Does every reflexive Banach space have the FPP?

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Problem 4:

Can Banach spaces with the FPP or the w -FPP be characterized by its intrinsic geometry?

Normal structure and the w -FPP

A Banach space has weakly normal structure (w -NS) if the following holds: for all convex weakly compact set $C \subset X$, there is some $x \in C$ and $0 < r < \text{diam}(C)$ such that

$$C \subset B(x, r); \quad r := \sup_{y \in C} \|x - y\| < \text{diam}(C)$$

Theorem (A. Kirk)

Every Banach space with w -NS has w -FPP



W. A. Kirk, *A Fixed Point Theorem for Mappings which do not Increase Distances*. Amer. Math. Monthly, 1965.

There are nonreflexive Banach spaces with w -NS: Every UCED has w -NS and every separable Banach space can be renormed to be UCED. Hence, every separable Banach space can be renormed to have the w -FPP.