Metric Fixed Point Theory

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Introduction to metric fixed point theory

Let (C, d) be a metric space and $T : C \to C$. T is L-Lipschitz if

$$d(Tx, Ty) \le Ld(x, y), \qquad \forall x, y \in C.$$

$x \in C$ is a fixed point for T if $T(x) = x$.

When can we assure the existence of a fixed point? It depends on the value of L and "metric" or "topological" properties of the domain C.

Theorem (Banach Contraction Principle)

Let (C, d) be a complete m.s. and $T : C \to C$ L-Lipschitz with $L < 1$ (contraction). Then there exists a unique fixed point.

Hilbert spaces and L-Lipschitz mappings with $L > 1$

For every $L > 1$, there is a L-Lipschitz mapping from the unit ball of a Hilbert space into itself without fixed points.

$$H = \ell_2 = \left\{ x = \{x_n\} \subset \mathbb{R} : \|x\|_2 := \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2} < \infty \right\}$$

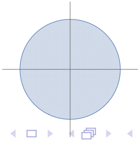
Given $\epsilon > 0$, define $T_\epsilon : B_H \to B_H$ by

$$T_\epsilon(x_1, x_2, \cdots) = (\epsilon(1 - \|x\|_2), x_1, x_2, \cdots)$$

Then

$$\|T_\epsilon x - T_\epsilon y\|_2 \leq \sqrt{1 + \epsilon^2}\ \|x - y\|_2; \quad \forall x, y \in B_H :$$

T_ϵ is $\sqrt{1+\epsilon^2}$-Lipschitzian and T_ϵ fails to have a fixed point.



Fixed-point free 1-Lipschitz mappings (nonexpansive)

$T : (\mathbb{R}, |\cdot|) \to (\mathbb{R}, |\cdot|)$, $T(x) = x + a$ $(a \neq 0)$ fixed-point free.

$c_0 = \{(x_n) \subset \mathbb{R} : \lim_n x_n = 0, \|x\|_\infty = \sup_n |x_n|\}$

Kakutani Counterexample

$$T : B_{c_0} \to B_{c_0};\; T(x_1, x_2, \cdots) = (1, x_1, x_2, \cdots)$$

is a fixed-point free isometry $\|Tx - Ty\|_\infty = \|x - y\|_\infty$, $x, y \in B_{c_0}$

$$\ell_p := \left\{ x := (t_n) : \|x\|_p = \left[\sum_{n=1}^{\infty} |t_n|^p \right]^{\frac{1}{p}} < +\infty \right\}, \quad 1 \leq p < +\infty$$

Another counter-example

$C = \{x := (t_n) \in \ell_1 : t_n \geq 0, \sum_{n=1}^{\infty} t_n = 1\}$ bounded, closed in ℓ_1. Let $T : C \to C$ be given by:

$$T(t_1, t_2, t_3, \cdots) = (0, t_1, t_2, t_3, \cdots)$$

T is fixed-point free and $\|Tx - Ty\|_1 = \|x - y\|_1$.

If a counterexample can be found in a Banach space X, then it can be lifted to every Banach space Y cointaining an isometric copy of X.

Approximate fixed point sequences

Let $T : C \to C$ be nonexpansive, C closed, convex bounded of a Banach space X. Then: $\inf\{\|Tx - x\| : x \in C\} = 0$.

A sequence $(x_n) \subset C$ such that $\lim_n \|x_n - Tx_n\| = 0$ is called an approximate fixed point sequence (afps).

Remark: Boundedness hypothesis cannot be dropped

A convex subset $C \subset X$ is has the approximate fixed point sequence (AFPP) if for all $T : C \to C$ nonexpansive, $\inf\{\|x - Tx\| : x \in C\} = 0$.

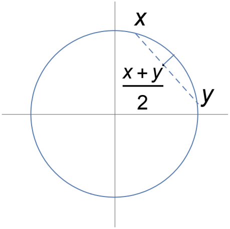
I. Shafrir, *The approximate fixed point property in Banach spaces and hyperbolic spaces.* Israel J. of Math. 1990.

A convex subset $C \subset X$ has the AFPP if and only if: for every sequence $(x_n) \subset C$ with $\|x_n\| \to \infty$ and for every $f \in S_{X^*}$, $\limsup_n f\left(\frac{x_n}{\|x_n\|}\right) < 1$.

Positive results

X is uniformly convex (UC) if: for very $\epsilon \in (0,2]$, $\exists\ \delta = \delta_X(\epsilon) > 0$:

$$\left.\begin{array}{l} \|x\| = 1 \\ \|y\| = 1 \\ \|x-y\| \geq \epsilon \end{array}\right\} \Longrightarrow \left\| \frac{x+y}{2} \right\| \leq 1 - \delta_X(\epsilon)$$



Equivalently:

$$\left.\begin{array}{l} \|x_n\| \to r \\ \|y_n\| \to r \\ \left\| \frac{x_n+y_n}{2} \right\| \to r \end{array}\right\} \Longrightarrow \|x_n - y_n\| \to 0.$$

Theorem (F. Browder, 1965)

Let $(X, \|\cdot\|)$ be a Hilbert space or, more generally, a Banach space uniformly convex (UC). Let $C \subset X$ be closed convex and bounded. Let $T : C \to C$ be nonexpansive. Then T has a fixed point.

Proof.

Every UC Banach space is reflexive (closed convex bounded sets are weakly compact).

Let (x_n) be an afps. For all $r > 0$,

$$\{z \in C : \limsup_n \|x_n - z\| \leq r\}$$

is closed convex bounded and T-invariant (and weakly compact)

$r_0 = r(\{x_n\}, C) := \inf\{\limsup_n \|x_n - z\| : z \in C\}$ (Asymp. radius)

$Z_0 = Z(\{x_n\}, C) = \{z \in C : \limsup_n \|x_n - z\| = r_0\}$ (Asymp. center)

$Z_0 = \bigcap_n \{z \in C : \limsup_n \|x_n - z\| \leq r_0 + \frac{1}{n}\}$.

$Z_0 \neq \emptyset$ (by weak-compactness) and T-invariant.
Z_0 is a singleton. Indeed, if $z_1, z_2 \in Z_0$, then
$\limsup_n \|x_n - z_1\| = \limsup_n \|x_n - z_2\| = \limsup_n \left\|x_n - \frac{z_1+z_2}{2}\right\| = r_0$,
then $\lim_n \|(x_n - z_1) - (x_n - z_2)\| = \|z_1 - z_2\| = 0$ and $z_1 = z_2 = Tz_1$.

$\square$

Lebesgue spaces

Let (Ω, Σ, μ) a σ-finite measure space, $1 \leq p < +\infty$.

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : \text{measurable } \|f\|_p = \left[\int_\Omega |f(t)|^p d\mu \right]^{1/p} < \infty \right\}$$

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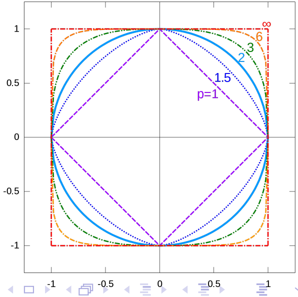
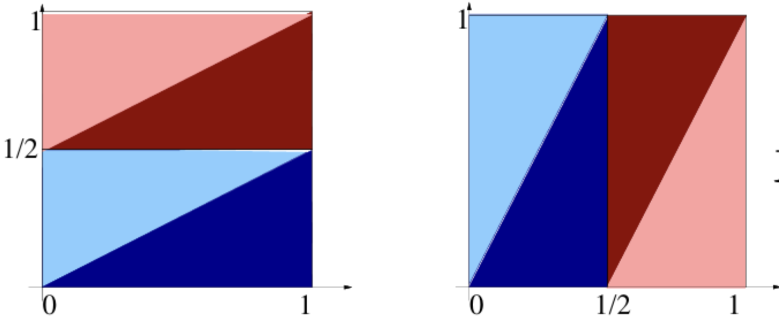
If $1 < p < +\infty$, $L^p(\Omega)$ is UC.

Weak-compactness is not enough.



The baker transform: An example of an ergodic mapping

$$S : [0,1) \to [0,1);\ S(x,y) = \begin{cases} \left(\frac{x}{2}, 2y\right) & 0 \le y \le \frac{1}{2} \\ \left(\frac{x+1}{2}, 2y-1\right) & \frac{1}{2} \le y < 1 \end{cases}$$



Weak-compactness for the domain is not enough

There exists a convex, weakly compact subset C in $L^1[0,1]$ and a fixed-point free isometry $T : C \to C$.

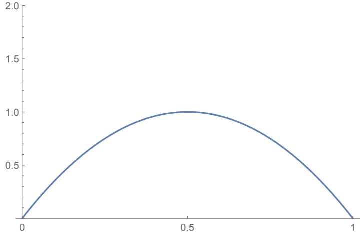
$C = \left\{ f : [0,1] \to [0,1], \text{ measurable}, \int_0^1 f\,dm = \frac{1}{2} \right\}$. $T : C \to C$,

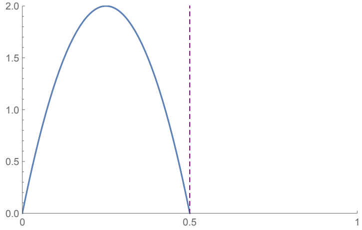
$$Tf(t) = \begin{cases} \min\{2f(2t), 1\} & 0 \le t \le \frac{1}{2} \\ \max\{2f(2t-1)-1, 0\} & \frac{1}{2} < t \le 1 \end{cases}$$

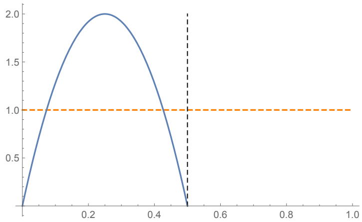
T is fixed point free in C and $\|Tf - Tg\|_1 = \|f - g\|_1 \ \forall f, g \in C$.

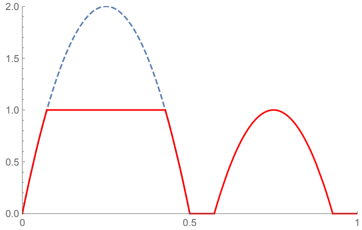
D. Alspach, *A fixed point free nonexpansive map*, Proc. Amer. Math. Soc., 1981.

Image of $f(t) = -4(t - \frac{1}{2})^2 + 1$ under Alspach's example









Definitions

A subset $C \subset X$ has the FPP if for every $T : C \to C$ nonexpansive there is a fixed point.

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Problem 1

Does there exist a closed convex unbounded set with the FPP?

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Definition

A Banach space X is said to have the FPP when every closed convex bounded set has the FPP.

A Banach space X is said to have the w-FPP when every convex weakly compact set has the FPP.

- c_0, ℓ_1 fail the FPP (and every Banach space containing an isometric copy of either c_0 or ℓ_1)
- $L_1[0,1]$ fails the w-FPP

A super-space for all separable Banach space

$\ell_\infty := \{x = (x_n) : (x_n)\text{bounded}, .\|x\|_\infty = \sup_n |x_n|\}$

Every separable B.s. is isometrically embedded in $(\ell_\infty, \|\cdot\|_\infty)$.

Take (x_n) dense in X and $x_n^* \in X^*$ with $x_n^*(x_n) = \|x_n\|$. Then

$$X \to \ell_\infty; \quad x \to (x_1^*(x), x_2^*(x), \cdots, x_n^*(x), \cdots) \text{ isometry, that is}$$

$$\|Tx - Ty\|_\infty = \|x - y\|_X, \quad \forall x, y \in X.$$

ℓ_∞ fails the w-FPP.

Problem 2

Does there exist a Banach space X failing the w-FPP and without isometric copies of $L_1[0,1]$?

UC $\Rightarrow$ FPP

UC implies reflexivity.

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Problem 3:

Does every reflexive Banach space have the FPP?

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Problem 4:

Can Banach spaces with the FPP or the w-FPP be characterized by its intrinsic geometry?

Normal structure and the w-FPP

A Banach space has weakly normal structure (w-NS) if the following holds: for all convex weakly compact set $C \subset X$ with $\mathrm{diam}(C) > 0$, there is some $x \in C$ and $0 < r < diam(C)$ such that

$$C \subset B(x, r); \qquad r := \sup_{y \in C} \|x - y\| < diam(C)$$

Theorem (A. Kirk)

Every Banach space with w-NS has w-FPP

W. A. Kirk, *A Fixed Point Theorem for Mappings which do not Increase Distances*. Amer. Math. Monthly, 1965.

There are nonreflexive Banach spaces with w-NS: Every UCED has w-NS and every separable Banach space can be renormed to be UCED.Hence, every separable Banach space can be renormed to have the w-FPP.

Homework

- Every separable Banach space can be renormed to have the w-FPP.
- ℓ_∞ can be renormed to be UCED, so ℓ_∞ can be renormed to have the w-FPP.
- ℓ_∞ cannot be renormed to have the FPP.

  P.N. Dowling, C.J. Lennard, B. Turett *Reflexivity and the fixed-point property for nonexpansive maps* J. Math. Anal. Appl., 1996.

- For Γ uncountable, every renorming of $\ell_\infty(\Gamma)$ contains an isometric copy of ℓ_∞ (J. R. Partington).

  $\ell_\infty(\Gamma)$ cannot be renormed to have the w-FPP.

The lack of fixed points determines bizarre weakly compact domains

$T : C \to C$ nonexpansive, C convex, weakly-compact, $Fix(T) = \emptyset$.

- We can assume that C is minimal T-invariant (Zorn Lemma).
- $C = \overline{co}T(C)$.
- Define $r(C) = \inf\{\sup_{y\in C} \|x - y\| : x \in C\}$ (Chebyshev radius) Chebyshev center:

$$Z(C) = \{x \in C : \sup_{y\in C} \|x - y\| = r(C)\}$$

is nonempty, convex and weakly compact and T-invariant.

- $Z(C) = C$ and then $r(C) = \text{diam}(C)$ (C is diametral).

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  is nonempty, convex and weakly compact and T-invariant.
- $Z(C) = C$ and then $r(C) = \mathrm{diam}(C)$ (C is diametral).

Under the umbrella of normal structure, $diam(C) = 0$ and $C = \{x\}$ a fixed point (and this is the proof of Kirk's result)

w-NS is not necessary for the w-FPP

The Banach space c_0 fails to have w-NS:

Take $C = \overline{co}(e_n)$. $\mathrm{diam}(C) = 1$ and for all $x \in co(e_n)$, $\sup_{y \in C} \|x - y\| = 1$.

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Theorem (Maurey, 1980)

c_0 has the w-FPP.

B. Maurey, *Points fires des contractions sur un convex firme de L_1*. Seminaire d'Analyse Fonctionelle, 1980-81, Expose num. VIII, Ecole Polytechnique, 1981.

(Improving a previous result $C = \overline{co}(x_n)$, (x_n) w-convergent.
E. Odell, Y. Sternfeld. *A fixed point theorem in c_0*, Pacific J. of Math. 1981)

The lack of fixed points determines bizarre weakly compact domains

Let $T : C \to C$ be nonexpansive, $Fix(T) = \emptyset$, with C convex weakly compact and minimal. Take $(x_n) \subset C$ an afps.

- $Z(\{x_n\}, C) = C$

  Indeed: $C_r := \{z \in C : \limsup_n \|x_n - z\| \le r\} \subset C$ is either empty or convex, w-comapact and T-invariant. Then $C = C_r$, so there is only one possible $r > 0$ where

  $$\limsup_n \|x_n - z\| = r, \quad \forall z \in C.$$

- $\limsup_n \|x_n - z\| = \mathrm{diam}(C) \ \forall z \in C$ (Goebel-Karlovitz Lemma)

The quotient space $\ell_\infty(X)/c_0(X)$

$\tilde{X} := \ell_\infty(X)/c_0(X)$, $\tilde{x} = [(x_n)] \in \tilde{X}$, $\|\tilde{x}\| = \limsup_n \|x_n\|$.

$T : C \to C$, $Fix(T) = \emptyset$, C convex, w-compact, minimal.
WLOG: $\mathrm{diam}(C) = 1$, $(x_n) \subset C$ afps, $w - \lim_n x_n = 0 \in C$.

$\tilde{C} := \{\tilde{x} : \tilde{x} = [(x_n)] : x_n \in C \ \forall n \in \mathbb{N}\}$
$\tilde{T} : \tilde{C} \to \tilde{C}$ by $\tilde{T}([(x_n)]) = [(Tx_n)]$ is well-defined and nonexpansive.

Lemma (P.K. Lin's Lemma)

If $\tilde{W} \subset \tilde{C}$ is closed, convex and $\tilde{T} : \tilde{W} \to \tilde{W}$, then

$$\sup\{\|\tilde{w}\| : \tilde{w} \in \tilde{W}\} = 1 (= \mathrm{diam}(C))$$

Idea of Proof: By contradiction: $\sup\{\|\tilde{w}\| : \tilde{w} \in \tilde{W}\} \leq 1 - \epsilon$ for $\epsilon > 0$.
Take $(\tilde{w}^k)_k \subset \tilde{W}$ and afps for $\tilde{T}$. Every $\tilde{w}^k = [(w_n^k)_n]$.
We can construct $(w_{n_s}^{k_s})_s \subset C$ an afps for T and such that
$\limsup_s \|w_{n_s}^{k_s}\| \leq 1 - \epsilon$, which contradicts Goebel-Karlovitz Lemma.

Proof that c_0 has the w-FPP: (J. García-Falset)

Notice that if $(x_n) \subset c_0$ is weakly null and $x \in c_0$, then $\limsup_n \|x_n + x\| = \max\{\limsup_n \|x_n\|, \|x\|\}$.

By contradiction:

$T : C \to C$ as before and $\tilde{T} : \tilde{C} \to \tilde{C}$ in the quotient space $\tilde{X}$.
Let $(x_n) \subset C$ be a weakly null afps.

Define $\tilde{W} = \{\tilde{w} \in \tilde{C} : \|\tilde{w} - \tilde{x}\| \leq \frac{1}{2},\ \limsup_n \limsup_n \|w_n - w_m\| \leq \frac{1}{2}\}$

$\tilde{W}$ is closed, convex and $\tilde{T}$-invariant. Let $\tilde{w} = [(w_n)] \in \tilde{W}$. We can assume $\lim_n w_n = w$ weakly.
Then

$$\begin{aligned} \|\tilde{w}\| &= \limsup_n \|w_n\| = \limsup_n \|w_n - w + w\| \\ &= \max\{\limsup_n \|w_n - w\|, \|w\|\} \leq \tfrac{1}{2} < 1. \end{aligned}$$

and this is a contradiction with P.K. Lin's Lemma.

Reflexivity and FPP

Problem 3:

Does every reflexive Banach space have the FPP?

T. Domínguez-Benavides, *A renorming of some nonseparable Banach spaces with the fixed point property.* J. Math. Anal. Appl. 350, 525–530 (2009)

Theorem

Assume that there is a 1-to-1 linear continuous mapping $J : X \to c_0(\Gamma)$. Then X can be renormed to have the w-FPP.

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Idea of the proof:

Consider the renorming $|x| := \left(\|x\|_X^2 + \|J(x)\|_{c_0(\Gamma)}^2\right)^{\frac{1}{2}}$ and apply similar arguments as above.

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Corollary

If X is reflexive, then X can be renormed to have the FPP.