Metric Fixed Point Theory

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Introduction to metric fixed point theory

Let (C, d) be a metric space and $T : C \to C$. T is L-Lipschitz if

$$d(Tx, Ty) \leq Ld(x, y), \qquad \forall x, y \in C.$$

$x \in C$ is a fixed point for T if $T(x) = x$.

When can we assure the existence of a fixed point? It depends on the value of L and "metric" or "topological" properties of the domain C.

Theorem (Banach Contraction Principle)

Let (C, d) be a complete m.s. and $T : C \to C$ L-Lipschitz with $L < 1$ (contraction). Then there exists a unique fixed point.

Hilbert spaces and L-Lipschitz mappings with $L > 1$

For every $L > 1$, there is a L-Lipschitz mapping from the unit ball of a Hilbert space into itself without fixed points.

$$H = \ell_2 = \left\{ x = \{x_n\} \subset \mathbb{R} : \|x\|_2 := \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2} < \infty \right\}$$

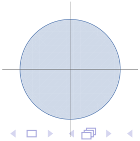
Given $\epsilon > 0$, define $T_\epsilon : B_H \to B_H$ by

$$T_\epsilon(x_1, x_2, \cdots) = (\epsilon(1 - \|x\|_2), x_1, x_2, \cdots)$$

Then

$$\|T_\epsilon x - T_\epsilon y\|_2 \leq \sqrt{1 + \epsilon^2}\, \|x - y\|_2; \quad \forall x, y \in B_H :$$

T_ϵ is $\sqrt{1+\epsilon^2}$-Lipschitzian and T_ϵ fails to have a fixed point.



Fixed-point free 1-Lipschitz mappings (nonexpansive)

$T : (\mathbb{R}, |\cdot|) \to (\mathbb{R}, |\cdot|)$, $T(x) = x + a$ $(a \neq 0)$ fixed-point free.

$c_0 = \{(x_n) \subset \mathbb{R} : \lim_n x_n = 0, \|x\|_\infty = \sup_n |x_n|\}$

Kakutani Counterexample

$$T : B_{c_0} \to B_{c_0};\ T(x_1, x_2, \cdots) = (1, x_1, x_2, \cdots)$$

is a fixed-point free isometry $\|Tx - Ty\|_\infty = \|x - y\|_\infty$, $x, y \in B_{c_0}$

$$\ell_p := \left\{ x := (t_n) : \|x\|_p = \left[\sum_{n=1}^{\infty} |t_n|^p \right]^{\frac{1}{p}} < +\infty \right\}, \quad 1 \leq p < +\infty$$

Another counter-example

$C = \{x := (t_n) \in \ell_1 : t_n \geq 0, \sum_{n=1}^{\infty} t_n = 1\}$ bounded, closed in ℓ_1. Let $T : C \to C$ be given by:

$$T(t_1, t_2, t_3, \cdots) = (0, t_1, t_2, t_3, \cdots)$$

T is fixed-point free and $\|Tx - Ty\|_1 = \|x - y\|_1$.

If a counterexample can be found in a Banach space X, then it can be lifted to every Banach space Y cointaining an isometric copy of X.

Approximate fixed point sequences

Let $T : C \to C$ be nonexpansive, C closed, convex bounded of a Banach space X. Then: $\inf\{\|Tx - x\| : x \in C\} = 0$.

A sequence $(x_n) \subset C$ such that $\lim_n \|x_n - Tx_n\| = 0$ is called an approximate fixed point sequence (afps).

Remark: Boundedness hypothesis cannot be dropped

A convex subset $C \subset X$ is has the approximate fixed point sequence (AFPP) if for all $T : C \to C$ nonexpansive, $\inf\{\|x - Tx\| : x \in C\} = 0$.

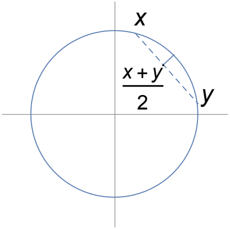
I. Shafrir, *The approximate fixed point property in Banach spaces and hyperbolic spaces.* Israel J. of Math. 1990.

A convex subset $C \subset X$ has the AFPP if and only if: for every sequence $(x_n) \subset C$ with $\|x_n\| \to \infty$ and for every $f \in S_{X^*}$, $\limsup_n f\left(\frac{x_n}{\|x_n\|}\right) < 1$.

Positive results

X is uniformly convex (UC) if: for very $\epsilon \in (0,2]$, $\exists\ \delta = \delta_X(\epsilon) > 0$:

$$\left.\begin{array}{l} \|x\| = 1 \\ \|y\| = 1 \\ \|x-y\| \geq \epsilon \end{array}\right\} \Longrightarrow \left\|\frac{x+y}{2}\right\| \leq 1 - \delta_X(\epsilon)$$



Equivalently:

$$\left.\begin{array}{l} \|x_n\| \to r \\ \|y_n\| \to r \\ \left\|\frac{x_n+y_n}{2}\right\| \to r \end{array}\right\} \Longrightarrow \|x_n - y_n\| \to 0.$$

Theorem (F. Browder, 1965)

Let $(X, \|\cdot\|)$ be a Hilbert space or, more generally, a Banach space uniformly convex (UC). Let $C \subset X$ be closed convex and bounded. Let $T : C \to C$ be nonexpansive. Then T has a fixed point.

Proof.

Every UC Banach space is reflexive (closed convex bounded sets are weakly compact).

Let (x_n) be an afps. For all $r > 0$,

$$\{z \in C : \limsup_n \|x_n - z\| \le r\}$$

is closed convex bounded and T-invariant (and weakly compact)

$r_0 = r(\{x_n\}, C) := \inf\{\limsup_n \|x_n - z\| : z \in C\}$ (Asymp. radius)

$Z_0 = Z(\{x_n\}, C) = \{z \in C : \limsup_n \|x_n - z\| = r_0\}$ (Asymp. center)

$Z_0 = \bigcap_n \{z \in C : \limsup_n \|x_n - z\| \le r_0 + \frac{1}{n}\}$.

$Z_0 \neq \emptyset$ (by weak-compactness) and T-invariant.
Z_0 is a singleton. Indeed, if $z_1, z_2 \in Z_0$, then
$\limsup_n \|x_n - z_1\| = \limsup_n \|x_n - z_2\| = \limsup_n \left\|x_n - \frac{z_1+z_2}{2}\right\| = r_0$,
then $\lim_n \|(x_n - z_1) - (x_n - z_2)\| = \|z_1 - z_2\| = 0$ and $z_1 = z_2 = Tz_1$. $\square$

Lebesgue spaces

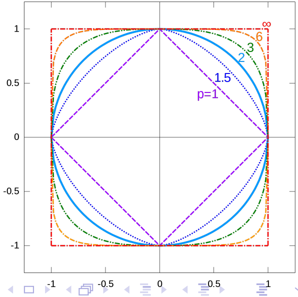
Let (Ω, Σ, μ) a σ-finite measure space, $1 \leq p < +\infty$.

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : \text{measurable } \|f\|_p = \left[\int_\Omega |f(t)|^p d\mu \right]^{1/p} < \infty \right\}$$

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R} : \text{measurable } \|f\|_\infty = \operatorname*{ess\,sup}_{t \in \Omega} |f(t)| < \infty \right\}$$

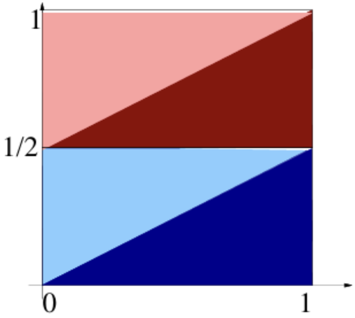
If $1 < p < +\infty$, $L^p(\Omega)$ is UC.

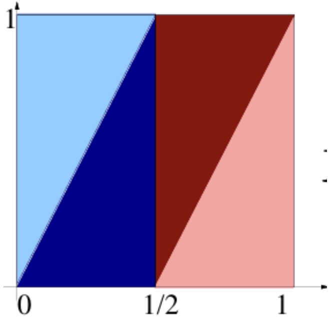
Weak-compactness is not enough.



The baker transform: An example of an ergodic mapping

$$S : [0,1) \to [0,1);\ S(x,y) = \begin{cases} \left(\frac{x}{2}, 2y\right) & 0 \le y \le \frac{1}{2} \\ \left(\frac{x+1}{2}, 2y-1\right) & \frac{1}{2} \le y < 1 \end{cases}$$





Weak-compactness for the domain is not enough

There exists a convex, weakly compact subset C in $L^1[0,1]$ and a fixed-point free isometry $T : C \to C$.

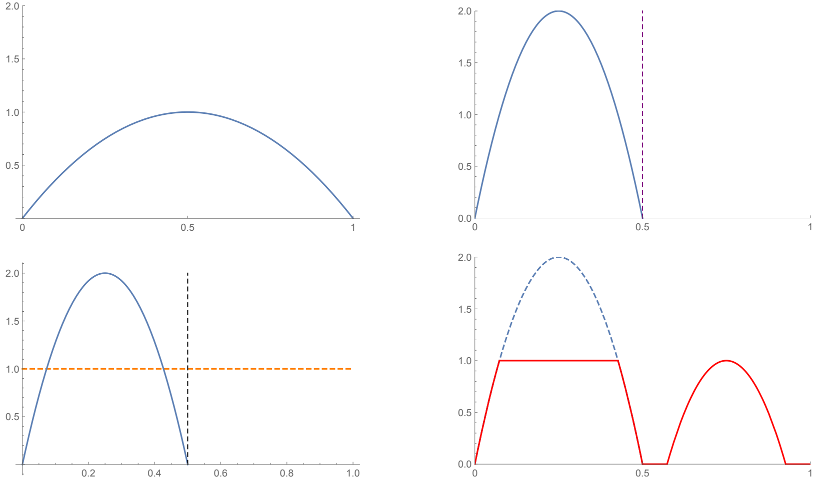
$C = \left\{ f : [0,1] \to [0,1], \text{ measurable}, \int_0^1 f\,dm = \frac{1}{2} \right\}$. $T : C \to C$,

$$Tf(t) = \begin{cases} \min\{2f(2t), 1\} & 0 \le t \le \frac{1}{2} \\ \max\{2f(2t-1) - 1, 0\} & \frac{1}{2} < t \le 1 \end{cases}$$

T is fixed point free in C and $\|Tf - Tg\|_1 = \|f - g\|_1 \ \forall f, g \in C$.

D. Alspach, *A fixed point free nonexpansive map*, Proc. Amer. Math. Soc., 1981.

Image of $f(t) = -4(t - \frac{1}{2})^2 + 1$ under Alspach's example



Definitions

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Problem 1

Does there exist a closed convex unbounded set with the FPP?

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Definition

A Banach space X is said to have the FPP when every closed convex bounded set has the FPP.

A Banach space X is said to have the w-FPP when every convex weakly compact set has the FPP.

- c_0, ℓ_1 fail the FPP (and every Banach space containing an isometric copy of either c_0 or ℓ_1)
- $L_1[0, 1]$ fails the w-FPP

A super-space for all separable Banach space

$\ell_\infty := \{x = (x_n) : (x_n)\text{bounded}, .\|x\|_\infty = \sup_n |x_n|\}$

Every separable B.s. is isometrically embedded in $(\ell_\infty, \|\cdot\|_\infty)$.

Take (x_n) dense in X and $x_n^* \in X^*$ with $x_n^*(x_n) = \|x_n\|$. Then

$$X \to \ell_\infty; \quad x \to (x_1^*(x), x_2^*(x), \cdots, x_n^*(x), \cdots) \text{ isometry, that is}$$

$$\|Tx - Ty\|_\infty = \|x - y\|_X, \quad \forall x, y \in X.$$

ℓ_∞ fails the w-FPP.

Problem 2

Does there exist a Banach space X failing the w-FPP and without isometric copies of $L_1[0,1]$?

UC $\Rightarrow$ FPP
UC implies reflexivity.

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Problem 3:

Does every reflexive Banach space have the FPP?

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Problem 4:

Can Banach spaces with the FPP or the w-FPP be characterized by its intrinsic geometry?

Normal structure and the w-FPP

A Banach space has weakly normal structure (w-NS) if the following holds: for all convex weakly compact set $C \subset X$ with $\operatorname{diam}(C) > 0$, there is some $x \in C$ and $0 < r < diam(C)$ such that

$$C \subset B(x, r); \qquad r := \sup_{y \in C} \|x - y\| < diam(C)$$

Theorem (A. Kirk)

Every Banach space with w-NS has w-FPP

W. A. Kirk, *A Fixed Point Theorem for Mappings which do not Increase Distances*. Amer. Math. Monthly, 1965.

There are nonreflexive Banach spaces with w-NS: Every UCED has w-NS and every separable Banach space can be renormed to be UCED.Hence, every separable Banach space can be renormed to have the w-FPP.

Homework

- Every separable Banach space can be renormed to have the w-FPP.
- ℓ_∞ can be renormed to be UCED, so ℓ_∞ can be renormed to have the w-FPP.
- ℓ_∞ cannot be renormed to have the FPP.

  P.N. Dowling, C.J. Lennard, B. Turett *Reflexivity and the fixed-point property for nonexpansive maps* J. Math. Anal. Appl., 1996.

- For Γ uncountable, every renorming of $\ell_\infty(\Gamma)$ contains an isometric copy of ℓ_∞ (J. R. Partington).

  $\ell_\infty(\Gamma)$ cannot be renormed to have the w-FPP.

The lack of fixed points determines bizarre weakly compact domains

$T : C \to C$ nonexpansive, C convex, weakly-compact, $Fix(T) = \emptyset$.

- We can assume that C is minimal T-invariant (Zorn Lemma).
- $C = \overline{co}T(C)$.
- Define $r(C) = \inf\{\sup_{y\in C} \|x - y\| : x \in C\}$ (Chebyshev radius) Chebyshev center:

$$Z(C) = \{x \in C : \sup_{y\in C} \|x - y\| = r(C)\}$$

  is nonempty, convex and weakly compact and T-invariant.
- $Z(C) = C$ and then $r(C) = \mathrm{diam}(C)$ (C is diametral).

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- $Z(C) = C$ and then $r(C) = \text{diam}(C)$ (C is diametral).

Under the umbrella of normal structure, $diam(C) = 0$ and $C = \{x\}$ a fixed point (and this is the proof of Kirk's result)

w-NS is not necessary for the w-FPP

The Banach space c_0 fails to have w-NS:

Take $C = \overline{co}(e_n)$. $\operatorname{diam}(C) = 1$ and for all $x \in co(e_n)$, $\sup_{y \in C} \|x - y\| = 1$.

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Theorem (Maurey, 1980)

c_0 has the w-FPP.

B. Maurey, *Points fixes des contractions sur un convex firme de L_1*. Seminaire d'Analyse Fonctionelle, 1980-81, Expose num. VIII, Ecole Polytechnique, 1981.

(Improving a previous result $C = \overline{co}(x_n)$, (x_n) w-convergent.

E. Odell, Y. Sternfeld. *A fixed point theorem in c_0*, Pacific J. of Math. 1981)

The lack of fixed points determines bizarre weakly compact domains

Let $T : C \to C$ be nonexpansive, $Fix(T) = \emptyset$, with C convex weakly compact and minimal. Take $(x_n) \subset C$ an afps.

- $Z(\{x_n\}, C) = C$

  Indeed: $C_r := \{z \in C : \limsup_n \|x_n - z\| \leq r\} \subset C$ is either empty or convex, w-comapact and T-invariant. Then $C = C_r$, so there is only one possible $r > 0$ where

  $$\limsup_n \|x_n - z\| = r, \quad \forall z \in C.$$

- $\limsup_n \|x_n - z\| = \mathrm{diam}(C)\ \forall z \in C$ (Goebel-Karlovitz Lemma)

The quotient space $\ell_\infty(X)/c_0(X)$

$\tilde{X} := \ell_\infty(X)/c_0(X)$, $\tilde{x} = [(x_n)] \in \tilde{X}$, $\|\tilde{x}\| = \limsup_n \|x_n\|$.

$T : C \to C$, $Fix(T) = \emptyset$, C convex, w-compact, minimal.
WLOG: $\mathrm{diam}(C) = 1$, $(x_n) \subset C$ afps, $w - \lim_n x_n = 0 \in C$.

$\tilde{C} := \{\tilde{x} : \tilde{x} = [(x_n)] : x_n \in C \ \forall n \in \mathbb{N}\}$
$\tilde{T} : \tilde{C} \to \tilde{C}$ by $\tilde{T}([(x_n)]) = [(Tx_n)]$ is well-defined and nonexpansive.

Lemma (P.K. Lin's Lemma)

If $\tilde{W} \subset \tilde{C}$ is closed, convex and $\tilde{T} : \tilde{W} \to \tilde{W}$, then

$$\sup\{\|\tilde{w}\| : \tilde{w} \in \tilde{W}\} = 1 (= \mathrm{diam}(C))$$

Idea of Proof: By contradiction: $\sup\{\|\tilde{w}\| : \tilde{w} \in \tilde{W}\} \leq 1 - \epsilon$ for $\epsilon > 0$.
Take $(\tilde{w}^k)_k \subset \tilde{W}$ and afps for $\tilde{T}$. Every $\tilde{w}^k = [(w_n^k)_n]$.
We can construct $(w_{n_s}^{k_s})_s \subset C$ an afps for T and such that $\limsup_s \|w_{n_s}^{k_s}\| \leq 1 - \epsilon$, which contradicts Goebel-Karlovitz Lemma.

Proof that c_0 has the w-FPP: (J. García-Falset)

Notice that if $(x_n) \subset c_0$ is weakly null and $x \in c_0$, then $\limsup_n \|x_n + x\| = \max\{\limsup_n \|x_n\|, \|x\|\}$.

By contradiction:

$T : C \to C$ as before and $\tilde{T} : \tilde{C} \to \tilde{C}$ in the quotient space $\tilde{X}$.
Let $(x_n) \subset C$ be a weakly null afps.

Define $\tilde{W} = \{\tilde{w} \in \tilde{C} : \|\tilde{w} - \tilde{x}\| \leq \frac{1}{2},\ \limsup_n \limsup_n \|w_n - w_m\| \leq \dfrac{1}{2}\}$

$\tilde{W}$ is closed, convex and $\tilde{T}$-invariant. Let $\tilde{w} = [(w_n)] \in \tilde{W}$. We can assume $\lim_n w_n = w$ weakly.
Then

$$\begin{aligned} \|\tilde{w}\| &= \limsup_n \|w_n\| = \limsup_n \|w_n - w + w\| \\ &= \max\{\limsup_n \|w_n - w\|, \|w\|\} \leq \tfrac{1}{2} < 1. \end{aligned}$$

and this is a contradiction with P.K. Lin's Lemma.

Reflexivity and FPP

Problem 3:

Does every reflexive Banach space have the FPP?

T. Domínguez-Benavides, *A renorming of some nonseparable Banach spaces with the fixed point property.* J. Math. Anal. Appl. 350, 525–530 (2009)

Theorem

Assume that there is a 1-to-1 linear continuous mapping $J : X \to c_0(\Gamma)$. Then X can be renormed to have the w-FPP.

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Idea of the proof:

Consider the renorming $|x| := \left(\|x\|_X^2 + \|J(x)\|_{c_0(\Gamma)}^2\right)^{\frac{1}{2}}$ and apply similar arguments as above.

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Corollary

If X is reflexive, then X can be renormed to have the FPP.

FPP beyond reflexivity

FPP does not imply reflexivity:

- Asympotically isometric copies of ℓ_1 and how the theory was pushed.
- A renorming of ℓ_1 with the FPP.
- More renormings of ℓ_1 with the FPP and new constructions of nonreflexive Banach spaces with the FPP (with "nice" Schauder basis and hereditarily ℓ_1)
- Nonreflexive function spaces with FPP: Variable Lebesgue spaces.

Definition (J. Hagler, 1972)

A Banach space $(X, \|\cdot\|)$ contains an asymptotically isometric copy of ℓ_1 if there exist a sequence $(x_n) \subset X$ and a sequence $(\epsilon_n) \downarrow 0$ such that

$$\sum_n (1-\epsilon_n)|t_n| \leq \left\| \sum_n t_n x_n \right\| \leq \sum_n |t_n| \qquad \forall (t_n) \in \ell_1 .$$

Theorem (P. Dowling, C. Lennard, B. Turett, 1996)

If a Banach space X contains an asymptotically isometric copy (a.i.c.) of ℓ_1, then X fails to have the FPP.

If Γ is uncountable, the Banach spaces $\ell_1(\Gamma)$ are not FPP-renormable.
The Banach space ℓ_∞ is not FPP-renormable.

Theorem (B. Maurey (1981), P. Dowling, C. Lennard, B. Turett (1996))

$X \subset L_1[0,1]$ es reflexive if and only if X has the FPP.

There exist some renormings of ℓ_1 without a.i.c. of ℓ_1

P.N.Dowling,C.J.Lennard,B.Turett, *Renormings of ℓ_1 and c_0 and fixed poin tproperties*, in:Handbook of Metric Fixed Point Theory, KluwerAcad.Publ., 2001.

Let $(\gamma_n) \subset (0,1)$ such that $\gamma_n \leq \gamma_{n+1} \uparrow 1$. Then

$$|||x||| := \sup_n \gamma_n \sum_{k=n}^{\infty} |a_k|, \qquad x = \sum_{n=1}^{\infty} a_n e_n$$

is an equivalent norm in ℓ_1 which does not have any a.i.c. of ℓ_1.

$$\gamma_1 \|x\|_1 \leq |||x||| \leq \|x\|_1 \qquad \forall x \in X.$$

P. N. Dowling, W.B. Johnson, C. Lennard, B. Turett, *The optimality of James's distortion theorems*. Proc. Amer. Math. Soc., 1997.

Let $p = (p_n) \subset (1 + \infty)$ converging to 1. Define

$$X = \mathbb{R} \oplus_{p_1} (\mathbb{R} \oplus_{p_2} (\mathbb{R} \oplus_{p_3} \mathbb{R} \oplus \dots))$$

$$v_1(x) = (|x_1|^{p_1} + |x_2|^{p_1})^{1/p_1}\,; \quad v_2(x) = \left(|x_1|^{p_1} + (|x_2|^{p_2} + |x_3|^{p_2})^{\frac{p_1}{p_2}}\right)^{1/p_1}$$

$$v_3(x) = \left(|x_1|^{p_1} + \left(|x_2|^{p_2} + (|x_3|^{p_3} + |x_4|^{p_3})^{\frac{p_2}{p_3}}\right)^{\frac{p_1}{p_2}}\right)^{1/p_1}, \dots$$

Here $v_p(x) = \lim_n v_n(x)$.

Under certain conditions over (p_n) (for instance for $p_n = \frac{2^n}{2^n - 1}$), the space $(X, v_p(\cdot))$ is isomorphic to ℓ_1.

$(X, v_p(\cdot))$ does not contain a.i.c. of ℓ_1.

P.K. Lin, *There is an equivalent norm on ℓ_1 that has the fixed point property*, Nonlinear Anal., 2008.

Theorem

$(\ell_1, |||\cdot|||)$ has the FPP.

$$|||x||| := \sup_n \gamma_n \sum_{k=n}^{\infty} |a_k|, \qquad x = \sum_n^{\infty} a_n e_n$$

There are non-reflexive Banach spaces with the FPP.

Carlos Hernández-Linares, Maria A. Japón. *A renorming in some Banach spaces with applications to fixed point theory* . J. Funct. Anal., 2010.

There exist some nonreflexive and non-isomorphic-to-ℓ_1 subspaces of $L_1[0,1]$ and some sequence Banach spaces that can be renormed to have the FPP (all of them are "similar" to ℓ_1)

E. Castillo-Santos, P. Dowling, H. Fetter, M. Japón, , C. Lennard, B. Sims, B. Turett, *Near-infinity concentrated norms and the fixed point property for nonexpansive maps on closed, bounded, convex sets.* J. Funct. Anal. 275 (2018)

$$\mathcal{A} = \{(x_n) \subset S_X : \sigma(\ell_1, c_0) - \lim_n x_n = 0\}$$

Theorem

A Banach space X have the FPP whenever:

X has a boundedly complete premonotone Schauder basis and, if (P_k) denotes the standard projections for the Schauder basis, the following hold:

- *For all $\epsilon > 0$ there exists some $k \in \mathbb{N}$ s.t. for $(x_n) \in \mathcal{A}$ and $P_k(x) = 0$:*

$$\limsup_n \|x_n + x\| \geq (1 - \epsilon) \left[\limsup_n \|x_n\| + \|x\|\right]$$

- *For all k, there exists a function $F_k : (0, 1) \to [0, +\infty)$ with $\limsup_{\lambda \to 0+} \frac{F(\lambda)}{\lambda} = 0$ such that the following holds:*

  When $(x_n) \in \mathcal{A}$, $P_k(x) = x$ and $\|x\| = 1$

$$\limsup_n \|x_n + \lambda x\| \leq \limsup_n \|x_n\| + F(\lambda)\|x\|.$$

The method could be applied to deduce P.K. Lin's result and some more:

Corollary

Let $p = (p_n) \subset (1 + \infty)$ *converging to 1. For*

$$X = \mathbb{R} \oplus_{p_1} (\mathbb{R} \oplus_{p_2} (\mathbb{R} \oplus_{p_3} \mathbb{R} \oplus \ldots))$$

$(X, v_p(\cdot) := \lim_n v_n(\cdot))$ *has the FPP.*

To this point, every known Banach space X with the FPP was a sequence space with a nice boundedly complete Shauder basis (where bounded sequences are coordinate-wise convergent) and hereditarily ℓ_1.

Variable Lebesgue Spaces $L^{p(\cdot)}(\Omega)$

Let (Ω, Σ, μ) be a σ-finite space and

$$p : \Omega \to [1, +\infty) : t \to p(t) \text{ measurable function.}$$

We ask for the set of measurable functions $f : \Omega \to \mathbb{R}$ verifying:

$$\rho(f) = \int_{\Omega} |f(t)|^{p(t)} d\mu < +\infty$$

- This set does not have vector space structure, in general.

  Example: consider $p : [1, +\infty) \to \mathbb{R}$, $p(t) = t$.
  $f(t) = \frac{1}{2}$ satisfies $\rho(f) < +\infty$ but $\rho(2f) = +\infty$.

A quick overview

These spaces where introduced 1931 by Orlicz as an class or Orlicz spaces which are more general

$$\int_\Omega \Phi(|f|)d\mu < \infty$$

Theory in Variable Lebegue Spaces has become very active in the last two decades due to their applications to

- Fluid dynamics
- Image processing
- Variational Calculus

Mathscinet: 13 papers (till 2000); 175 papers (2000-2009), 793 papers (2010-2019); 735 papers (2020-2026);

Some monographs about Variable Lebesgue Spaces

- D. V. Cruz-Uribe, A. Fiorenza, Variable Lebesgue Spaces: Foundations and Harmonic Analysis. Birkhauser: Basel 2013.
- L. Diening, P. Harjulehto, P. Hasto, M. Ruzicka. Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics (2011).
- M. Ruzicka, Electrorheological Fluids: Modeling and Mathematical Theory. Lecture Notes in Mathematics, vol. 1748. Springer, Berlin (2000).

Variable Lebesgue Spaces $L^{p(\cdot)}(\Omega)$

Let (Ω, Σ, μ) be a σ-finite measurable space and

$$p : \Omega \to [1, +\infty] : t \to p(t) \text{ measurable function.}$$

$\Omega_1 = \{t \in \Omega : p(t) = 1\}$, $\Omega_\infty = \{t \in \Omega : p(t) = +\infty\}$.

$$\rho(f) = \int_{\Omega \setminus \Omega_\infty} |f(t)|^{p(t)} d\mu + \|f 1_{\Omega_\infty}\|_\infty$$

$$L^{p(\cdot)}(\Omega) := \{f \text{ measurable} : \exists \lambda > 0 : \rho(f/\lambda) < +\infty\}$$

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leq 1 \right\}$$

$(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a Banach space

If $p(t) \equiv p$ then $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)}) = (L_p(\Omega), \|\cdot\|_p)$.

The atomic-case: Nakano sequence spaces

Assume $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$.

Take $p = (p_n) \subset [1, +\infty)$,

$$\rho(x) = \sum_{n=1}^{\infty} |x(n)|^{p_n}$$

and

$$\|x\|_{(p_n)} = \inf \left\{ \lambda > 0 : \sum_{n=1}^{\infty} \left(\frac{|x(n)|}{\lambda} \right)^{p_n} \leq 1 \right\}.$$

$(\ell^{(p_n)}, \|x\|_{(p_n)})$: Nakano spaces.

If $p_n = p$ for all $n \in \mathbb{N}$, $(\ell^{(p_n)}, \|x\|_{(p_n)}) = (\ell_p, \| \cdot \|_p)$.

Theorem

Let $(p_n) \subset [1, +\infty)$*:*
$(\ell^{p_n}, \| \cdot \|_{(p_n)})$ *is reflexive* $\Leftrightarrow 1 < \liminf_n p_n \leq \limsup_n p_n < +\infty$.

The w-FPP for $L^{p(\cdot)}(\Omega)$

T. Domínguez-Benavides, M. J. *Fixed point properties and reflexivity in variable Lebesgue spaces.* J. Funct. Anal. 280 (2021).

$$p_- := \operatorname{ess} \inf_{t \in \Omega \setminus \Omega_1} p(t), \qquad p_+ := \operatorname{ess} \sup_{t \in \Omega \setminus \Omega_\infty} p(t)$$

Lemma

If $p_+ = +\infty$, $(\ell_\infty, \|\cdot\|_\infty)$ embeds isometrically in $L^{p(\cdot)}(\Omega)$.

Theorem (Lukes, Pick, Pokorny, 2011)

For a non-atomic measure, $1 < p_-$ and $p_+ < \infty$ if and only if $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is uniformly convex (and reflexive).

Characterization of the w-FPP in VLS

Theorem

Let (Ω, Σ, μ) be a σ-finite measure space and let $p : \Omega \to [1, +\infty]$ be measurable. TFAE:

a) $L^{p(\cdot)}(\Omega)$ *satisfies the w-FPP.*

b) $L^{p(\cdot)}(\Omega)$ *does not contain isometrically* $(L^1[0,1], \|\cdot\|_1)$.

The proof is based on the normal structure property.

Problem 2

Does there exist a Banach space X failing the w-FPP and without isometric copies of $L_1[0,1]$? (No in case that VLS are considered)

Theorem

Assume that $p_+ < \infty$*.*

Let X *be a reflexive subspace of* $L^{p(\cdot)}(\Omega)$*. Then,* X *satisfies the FPP (as it happens for subspace of* $L_1[0,1]$*).*

However, the converse does not hold, in contrast to what happens in $L_1[0,1]$: there are some nonreflexive subspaces in $L^{p(\cdot)}(\Omega)$ when $p_- = 1$ with the FPP:

Let $(p_n) \subset (1, +\infty)$, $\lim_n p_n = 1$. Then $(\ell^{(p_n)}, \|x\|_{(p_n)})$ has the FPP.

Question: Does the same result hold when $\liminf_n p_n = 1$ (and $\limsup p_n < +\infty$?)

Every nonreflexive subspace of $L^{p(\cdot)}(\Omega)$ has an isomorphic copy of ℓ_1. However, there are not asymptotically isometric copies of ℓ_1 in $L^{p(\cdot)}(\Omega)$ unless it is isometric.

Characterization of the FPP in VLS

$\Omega_1 = \{t : p(t) = 1\}$, $\Omega_\infty = \{t : p(t) = \infty\}$.

Lemma

Let (Ω, Σ, μ) be a σ-finite measure and $p : \Omega \to [1, +\infty]$ be measurable. Assume that one of the following conditions holds:

i) $p_+(\Omega \setminus \Omega_\infty) = +\infty$;

ii) *The set Ω_1 either contains a nonatomic set with positive measure or infinitely many atoms;*

iii) *The set Ω_∞ either contains a nonatomic set with positive measure or infinitely many atoms .*

Then $L^{p(\cdot)}(\Omega)$ fails to have the FPP.

Theorem (T. Domínguez, MJ, PK Lin, 2026 (Preprint))

The following are equivalent:

- $L^{p(\cdot)}(\Omega)$ *has the FPP*
- $p_+(\Omega \setminus \Omega_\infty) < \infty$ *and* $\Omega_1 \cup \Omega_\infty$ *contains at most finitely many atoms and negligible sets.*
- $L^{p(\cdot)}(\Omega)$ *does not contain an isometric copy of* ℓ_1.

Idea of the Proof:
Take $(\gamma_n) \subset (1, +\infty)$ with $\lim_n \gamma_n = 1$ and define

$$P_n(f) = f1_{\{\gamma_n \le p\} \cup \Omega_\infty}, \qquad \mathcal{P} = \{P_n\}$$

Define a convergence: $\mathcal{P} - \lim_k x_k = x$ is $\lim_k P_n(x_k) = P_n(x)$ $\forall n \in \mathbb{N}$.

We use that $L^{p(\cdot)}(\{\gamma_n < p\})$ is (almost)-UC in order to construct approximate fixed point sequence that are $\mathcal{P}$-convergent close to asymptotic centers.

Verify that similar properties to the above are satisfied.

Some Open problems

- Does there exist a closed convex unbounded set with the FPP?
- Does there exist a Banach space X failing the w-FPP and without isometric copies of $L_1[0,1]$?
- Can c_0 be renormed to have the FPP?
- Does every reflexive Banach space have the FPP?
- Can a Hilbert space be renormed to fail the FPP?
- Does every super-reflexive Banach space have the FPP? (they have the fpp for isometries: J. Elton, P.K. Lin, E. Odell, S. Szarek, Contemp. Math, 1983)

Thank you very much for all your attention.