The two-disjoint-copies property for compact spaces, homogeneity and connection with C_p-theory

JERZY KĄKOL

A. MICKIEWICZ UNIVERSITY, POZNAŃ

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Motivation: The Separable Quotient Problem

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What happens for $C_p(K)$?

Problem 2 (J.K.–Saxon, J.K–liwa, Banakh–J.K–liwa)

Does there exist a continuous open linear map $C_p(X) \longrightarrow E$, where X is infinite compact and $E \subset \mathbb{R}^\omega$ is dense?

The C_p-Version

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Equivalent formulation:

Problem 3

Does $C_p(X)$ admit an infinite-dimensional metrizable quotient?

Why Efimov Spaces?

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- No example is known in ZFC.
- Any negative answer to our problem would have to be related to Efimov-type compacta.
- If $C_p(X)$ does not have infnite-dimensional metrizable quotients, then X is Efimov.

Two Canonical Quotients

The spaces

$$(c_0)_p,\ (\ell_\infty)_p$$

are Banach spaces endowed with the pointwise topology.

- $(c_0)_p$ and $(\ell_\infty)_p$ are dense subspaces of $\mathbb{R}^\omega$.
- They play in C_p-theory a role analogous to c_0 and ℓ_∞ in Banach-space theory.

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- If K is non-scattered, then K maps continuously onto $[0, 1]$.
- What can be said about quotients of $C_p(K)$?

The Two-Disjoint-Copies Property

Definition 4 (J.K–Śliwa)

A topological space X has the *two-disjoint-copies property* (2DCP) if there exists a sequence

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of non-empty compact subsets of X such that each K_n contains two disjoint copies of K_{n+1}.

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Self-similarity inside compact spaces

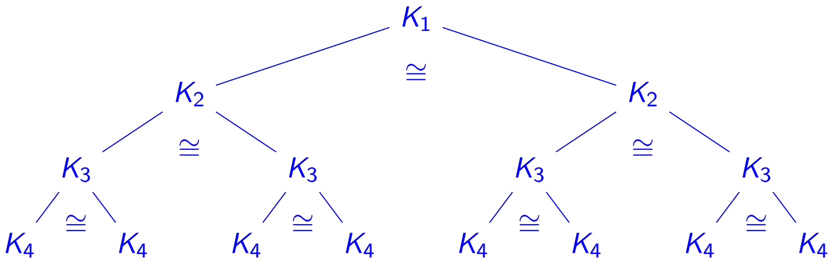

Figure: Ownership Scheme (2DCP). At each level, the compact sets occur in pairwise disjoint copies that are mutually homeomorphic (indicated by $\cong$) and homeomorphic to the sets constructed at the next level.

Theorem 5 (Banakh–J.K.–Śliwa)

If a Tychonoff space X has 2DCP, then $C_p(X)$ admits an infinite-dimensional metrizable quotient. The converse fails in ZFC.

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2DCP holds whenever:

1. X is locally homogeneous;
2. every non-empty open set contains two disjoint homeomorphic open subsets;
3. X contains two disjoint copies of itself;
4. X is extremally disconnected.

Homogeneity

Locally Homogeneous

if every $x, y \in X$ have open neighbourhoods V_x and V_y, such that there is a homeomorphism $\varphi : V_x \to V_y$ with $\varphi(x) = y$.

Homogeneous

if for $x, y \in X$ there is a homeomorphism $f : X \to X$ with $f(x) = y$.(**Examples**: topological groups, $[0,1]^\omega$; are not: $[0,1]^n$, $2^\omega \oplus \omega^\omega$, $\beta\omega \setminus \omega$, $\beta\omega$), **Rudin, Frolik, Kunen, van Douwen**...

h-Homogeneous

Every non-empty clopen subset of X is homeomorphic to X.
(**Examples**: connected spaces, $\mathbb{Q}$, 2^ω, ω^ω, $\beta\omega \setminus \omega$; are not: discrete spaces $|X| \geq 2$, $\omega \times 2^\omega$, $\beta\omega$). **van Mill, Terada, Medini, Carroy**...

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- locally homogeneous,
- has 2DCP,
- not homogeneous.

Reason:

$$\pi_1(S^2) = 0, \qquad \pi_1(T^2) = \mathbb{Z}^2.$$

Two Fundamental Quotient Theorems

Theorem 6 (Banakh–J.K.–Śliwa)

$C_p(X)$ has the JNP iff $C_p(X) \supset (c_0)_p$ as a complemented subspace iff there exists a continuous linear surjection $T : C_p(X) \to (c_0)_p$ iff $C_p(X)$ has a quotient isomorphic to $(c_0)_p$.

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Theorem 7 (Banakh–J.K.–Śliwa)

If $\beta\omega \subset X$, then $C_p(X)$ has a quotient isomorphic to $(\ell_\infty)_p$.

A Surprising ZFC Example

Theorem 8 (J.K.–Kurka–Śliwa)

Every topological space with 2DCP is uncountable and not scattered. There exists a perfect zero-dimensional compact space without 2DCP such that $C_p(K)$ contains a complemented copy of $(c_0)_p$.

Hence

$$2DCP \not\Longleftarrow (c_0)_p \text{ quotient.}$$

Corollary 9

A Rosenthal compact space X is scattered iff X does not have 2DCP.

Compact Metric Spaces

Theorem 10

For a compact metric space K we have: K has 2DCP iff K is uncountable iff $2^\omega \subset K$.

Example 11

The double arrow space $X = ([0,1] \times \{0,1\}) \setminus \{(0,0),(1,1)\}$ equipped with the order topology is

- compact,
- zero-dimensional,
- h-homogeneous,
- has 2DCP,
- contains neither $\beta\omega$ nor 2^ω.

Nevertheless, $C_p(X)$ has no quotient isomorphic to $(\ell_\infty)_p$ although has a complemented copy of $(c_0)_p$.

Efimov Spaces

Under additional set-theoretic assumptions:

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Under additional set-theoretic assumptions:

- Efimov spaces with 2DCP exist;
- Efimov spaces without 2DCP exist;
- no ZFC example is known.

A Consistent Counterexample

Problem 12

If a compact space X has 2DCP, must $C_p(X)$ admit a quotient isomorphic to $(c_0)_p$ or $(\ell_\infty)_p$?

Theorem 13 (J.K.–Kurka–Śliwa)

Consistently with ZFC, there exists a perfect zero-dimensional compact Efimov space K such that

1. K has 2DCP;
2. $C(K)$ is Grothendieck;
3. dens $C(K) = \omega_1 < \mathfrak{c}$;
4. $C_p(K)$ has no quotient isomorphic to $(c_0)_p$;
5. $C_p(K)$ has no quotient isomorphic to $(\ell_\infty)_p$.

Brech's Compact Space

Theorem 14 (Brech)

Assume CH.

Let $\kappa > \omega_1$ be regular and let G be $\mathbb{S}_\kappa$-generic, where $\mathbb{S}_\kappa$ is the product of κ copies of Sacks forcing.

If $B = P(\omega) \cap V$ and $K = S(B)$, then in $V[G]$ and $C(K)$ is a Grothendieck space and

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Moreover,

- K contains no copy of $\beta\omega$;
- K has no non-trivial convergent sequences;
- K nevertheless has 2DCP (by Theorem 13).

Further Consequences

- The same argument shows that Brech's space K has a basis of clopen sets each homeomorphic to K.
- In particular, K is h-homogeneous.
- Sobota and Zdomskyy studied preservation of the Nikodym property under Sacks forcing.
- If $\mathcal{A}$ is a σ-complete Boolean algebra in the ground model V, then $\mathcal{A}$ preserves the Nikodym property in the $\mathbb{S}_\kappa$-generic extension.
- The corresponding Stone space

$$K = S(\mathcal{A})$$

also admits 2DCP for $\mathcal{A} = P(\omega)$.

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4. The relationship between 2DCP, Efimov spaces and quotients of $C_p(X)$ remains largely open.

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1. 2DCP is a powerful structural property.
2. $$2DCP \implies C_p(X)$$
   has an infinite-dimensional metrizable quotient.
3. Homogeneity often implies 2DCP.
4. The relationship between 2DCP, Efimov spaces and quotients of $C_p(X)$ remains largely open.
5. There exist compact spaces with 2DCP which are not locally homogeneous without copies of $\beta\omega$ and 2^ω.

Thank you!