(λ^+)-injective Banach spaces

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The question

Let $\lambda \geqslant 1$. The notation

$$X \text{ is } (\lambda^+)\text{-injective}$$

means that X is $(\lambda + \varepsilon)$-injective for every $\varepsilon > 0$.

Quantitative extension problem

When does approximate extension with every constant above λ fail to give extension with constant λ?

$$(\lambda^+)\text{-injective} \quad \not\Longrightarrow \quad \lambda\text{-injective}.$$

The aim is to recover Pełczyński's theorem for all $\lambda > 1$.

λ-injectivity

A Banach space X is λ-injective if for every inclusion $E \subset Z$ and every bounded operator $T : E \to X$, there is an extension $\widetilde{T} : Z \to X$ satisfying

$$\widetilde{T}|_E = T, \qquad \|\widetilde{T}\| \leqslant \lambda \|T\|.$$

The case $\lambda = 1$ is the classical extension property.

$$\begin{tikzcd} E \arrow[r, "\iota"] \arrow[dr, "T"'] & Z \arrow[d, dashed, "\widetilde{T}"] \\ & X \end{tikzcd}$$

$$\|\widetilde{T}\| \leqslant \lambda \|T\|$$

Projection constants

For a closed subspace $E \subset F$, let

$$\mathcal{P}(F,E) = \{P : F \to E : P|_E = \mathrm{Id}_E\}, \qquad \lambda(E,F) = \inf_{P \in \mathcal{P}(F,E)} \|P\|.$$

Criterion

Let Z be 1-injective and let $Y \subset Z$ be closed. Put

$$a = \lambda(Y,Z) < \infty.$$

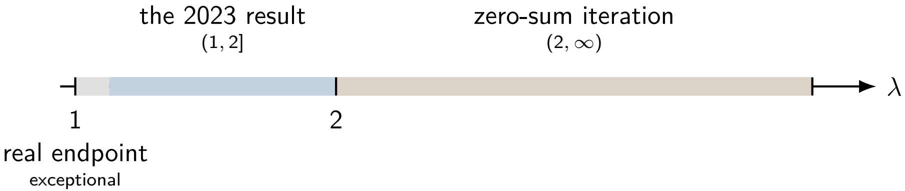
If the infimum in $\lambda(Y,Z)$ is not attained, then Y is (a^+)-injective but not a-injective.

Indeed, approximate projections $Z \to Y$ give approximate extensions; an a-extension of $\mathrm{Id}_Y : Y \to Y$ across $Y \subset Z$ would be a norm-a projection.

Pełczyński's theorem

Theorem A

For every $\lambda > 1$, there exists a Banach space which is (λ^+)-injective but not λ-injective.



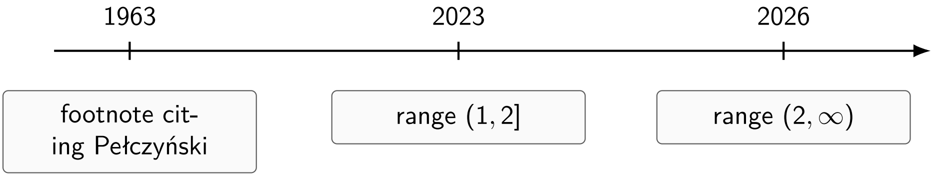
Historical note

Isbell and Semadeni proved that every infinite-dimensional 1-injective space contains a hyperplane which is $(2+\varepsilon)$-injective for every $\varepsilon > 0$, but not 2-injective.

In a footnote they recorded that Pełczyński had proved the corresponding assertion for every $\lambda > 1$.

The missing proof

No proof of Pełczyński's argument appears to have been preserved. The result therefore had to be reconstructed.



The range $1 < \lambda \leqslant 2$

The 2023 proof reconstructs Pełczyński's assertion in the hyperplane range, but it is not a recovery of his original argument.

For $1 < \lambda < 2$, one works in $X = \ell_\infty$ with

$$f = (1-b)g + bh_{a,n} \in \ell_\infty^* = \ell_1 \oplus_1 c_0^\perp, \qquad W_\lambda = \ker f,$$

where $g \in c_0^\perp$ is singular and non-norm-attaining. The hyperplane formulae give

$$\lambda(W_\lambda, \ell_\infty) = \lambda, \qquad \mathcal{P}(\ell_\infty, W_\lambda) \text{ contains no minimal projection.}$$

For $\lambda = 2$, a non-norm-attaining singular functional $g \in c_0^\perp$ gives

$$\lambda(\ker g, \ell_\infty) = 2, \qquad \ker g \text{ is not 2-injective.}$$

Later inputs

The proof uses results which were not available in the 1960s:

Blatter–Cheney (1974)	minimal projections on hyperplanes in sequence spaces,
Balcar–Franěk (1982)	independent families in complete Boolean algebras,
Baronti–Franchetti (1990)	minimal and polar projections onto hyperplanes in ℓ_p and ℓ_∞.

The endpoint $\lambda = 1$

Real scalars

Lindenstrauss proved

$$(1^+)\text{-injective} \quad \Longrightarrow \quad 1\text{-injective}$$

for real Banach spaces.

Complex scalars

It remains open whether

$$(1^+)\text{-injective} \quad \Longrightarrow \quad 1\text{-injective}$$

holds over $\mathbb{C}$.

A reduction

For the complex problem, I can reduce the question to proving that every complex (1^+)-injective space X has an extreme point in its unit ball:

$$\operatorname{ext} B_X \neq \varnothing.$$

With such an extreme point available, the argument can proceed towards 1-injectivity.

Why hyperplanes cannot suffice

The known construction in the range $(1, 2]$ uses hyperplanes in ℓ_∞.

For every Banach space Z and every hyperplane $H \subset Z$,

$$\lambda(H, Z) \leqslant 2.$$

Hence hyperplanes cannot produce relative projection constants larger than 2.

to reach $\lambda > 2$, pass
to finite-codimensional
subspaces in finite ℓ_∞-sums

The zero-sum construction

For a Banach space $X \neq \{0\}$ and $n \geqslant 2$, set

$$X_\infty^n = (X^n, \|(x_1, \dots, x_n)\|_\infty), \qquad \Sigma_n(X) = \left\{ (x_i)_{i=1}^n \in X_\infty^n : \sum_{i=1}^n x_i = 0 \right\}.$$

Define the centring projection

$$S_n^X(x_1, \dots, x_n) = \left(x_i - \frac{1}{n} \sum_{j=1}^n x_j \right)_{i=1}^n .$$

Elementary norm computation

$$S_n^X : X_\infty^n \to \Sigma_n(X) \quad \text{is a projection,} \qquad \|S_n^X\| = \mu_n := 2 - \frac{2}{n}.$$

Computing $\|S_n^X\|$

Let $M = \max_i \|x_i\|$. For each coordinate,

$$\left\| x_i - \frac{1}{n}\sum_{j=1}^{n} x_j \right\| = \left\| \frac{n-1}{n} x_i - \frac{1}{n}\sum_{j\neq i} x_j \right\| \leqslant \left(2 - \frac{2}{n}\right) M.$$

Thus $\|S_n^X\| \leqslant \mu_n$.

For the reverse inequality, choose $u \in S_X$ and put

$$x = (u, -u, \dots, -u) \in X_\infty^n.$$

Then $\|x\|_\infty = 1$ and the first coordinate of $S_n^X x$ is

$$u - \frac{2-n}{n} u = \left(2 - \frac{2}{n}\right) u.$$

Hence $\|S_n^X\| = \mu_n$.

The multiplication lemma

Let $Y \subset Z$, put $a = \lambda(Y, Z) < \infty$, and assume that the infimum defining $\lambda(Y, Z)$ is not attained. For

$$W = \Sigma_n(Y) \subset Z^n_\infty$$

one has

$$\boxed{\lambda(W, Z^n_\infty) = \mu_n\, a}, \qquad \mu_n = 2 - \frac{2}{n}.$$

Moreover, the infimum defining $\lambda(W, Z^n_\infty)$ is not attained.

$$\boxed{Y \subset Z} \longrightarrow \boxed{\Sigma_n(Y) \subset Z^n_\infty} \longrightarrow \boxed{\text{constant multiplied by } \mu_n}$$

Multiplication lemma: upper bound

Choose $Q_\varepsilon \in \mathcal{P}(Z, Y)$ with $\|Q_\varepsilon\| < a + \varepsilon$, and define the coordinatewise operator

$$\widehat{Q}_\varepsilon : Z_\infty^n \to Y_\infty^n, \qquad \widehat{Q}_\varepsilon(z_1, \dots, z_n) = (Q_\varepsilon z_1, \dots, Q_\varepsilon z_n).$$

Then

$$P_\varepsilon = S_n^Y \circ \widehat{Q}_\varepsilon : Z_\infty^n \longrightarrow \Sigma_n(Y)$$

is a projection onto $\Sigma_n(Y)$, and

$$\|P_\varepsilon\| \leqslant \|S_n^Y\| \, \|\widehat{Q}_\varepsilon\| < \mu_n(a + \varepsilon).$$

Consequently,

$$\lambda(\Sigma_n(Y), Z_\infty^n) \leqslant \mu_n a.$$

Multiplication lemma: symmetrisation

Let $P : Z_\infty^n \to \Sigma_n(Y)$ be any projection. For $\sigma \in \mathfrak{S}_n$, let U_σ permute the coordinates. Set

$$\widetilde{P} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} U_\sigma^{-1} P U_\sigma.$$

Then

$$\widetilde{P} \in \mathcal{P}(Z_\infty^n, \Sigma_n(Y)), \qquad \|\widetilde{P}\| \leqslant \|P\|,$$

and $\widetilde{P}$ commutes with all coordinate permutations.

Writing $e_j(z)$ for the vector with z in the j-th coordinate, symmetry gives operators $A, B : Z \to Y$ such that

$$\widetilde{P}(e_1 z) = (Az, Bz, \ldots, Bz).$$

Multiplication lemma: the lower bound

Since $\widetilde{P}(Z^n_\infty) \subset \Sigma_n(Y)$,

$$A + (n-1)B = 0.$$

Since $\widetilde{P}$ fixes $\Sigma_n(Y)$, applying it to $e_1(y) - e_2(y)$ gives

$$(A - B)y = y \qquad (y \in Y).$$

Thus $R := A - B \in \mathcal{P}(Z, Y)$, and solving the two equations yields

$$A = \frac{n-1}{n}R, \qquad B = -\frac{1}{n}R, \qquad \widetilde{P} = \widehat{R}\, S^Z_n.$$

Using the extremal vector for S^Z_n, with u chosen almost norm-attaining for R, gives

$$\|\widetilde{P}\| = \mu_n \|R\|.$$

By non-attainment, $\|R\| > a$; hence

$$\|P\| \geqslant \|\widetilde{P}\| > \mu_n a.$$

This proves both the lower bound and non-attainment at the new level.

Choosing the parameters for $\lambda > 2$

Fix $\lambda > 2$. Choose $m \geqslant 1$ with

$$2^m \leqslant \lambda < 2^{m+1}.$$

Since $\mu_N = 2 - 2/N \nearrow 2$, choose $N \geqslant 3$ so large that

$$\frac{\lambda}{2} < \mu_N^m < 2^m \leqslant \lambda.$$

Now put

$$\alpha = \frac{\lambda}{\mu_N^m}.$$

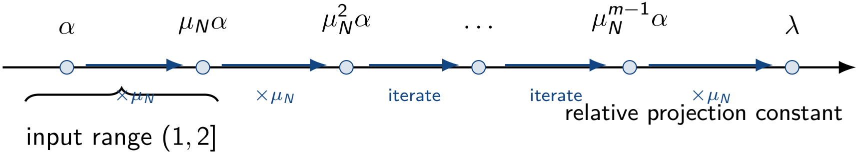
Then

$$1 < \alpha < 2,$$

so the starting constant is in the previously established range $(1, 2]$.

This is the reduction of the range $\lambda > 2$ to smaller constants.

The reduction in one diagram



At each step,

$$(Y_{k-1} \subset Z_{k-1}) \quad \longmapsto \quad (Y_k, Z_k) = (\Sigma_N(Y_{k-1}), (Z_{k-1})_\infty^N),$$

and the obstruction is multiplied by μ_N while non-attainment is preserved.

Iteration

Start with the $(1,2]$-case:

$$Y_0 \subset Z_0 := \ell_\infty, \qquad \lambda(Y_0, Z_0) = \alpha,$$

with non-attainment of the projection constant.

Define recursively, for $k = 1, \ldots, m$,

$$Z_k = (Z_{k-1})_\infty^N, \qquad Y_k = \Sigma_N(Y_{k-1}) \subset Z_k.$$

By induction from the multiplication lemma,

$$\lambda(Y_k, Z_k) = \mu_N^k \alpha, \qquad \lambda(Y_m, Z_m) = \mu_N^m \alpha = \lambda,$$

and non-attainment persists at every stage.

Conclusion of Theorem A for $\lambda > 2$

The spaces Z_k remain 1-injective:

$$Z_0 = \ell_\infty, \qquad Z_k = (Z_{k-1})_\infty^N,$$

and finite ℓ_∞-sums of 1-injective spaces are 1-injective.

Applying the projection-constant criterion to $Y_m \subset Z_m$,

$$\lambda(Y_m, Z_m) = \lambda$$

with non-attainment gives

$$Y_m \text{ is } (\lambda^+)\text{-injective but not } \lambda\text{-injective.}$$

Together with the $(1, 2]$-case, this establishes Pełczyński's theorem for every $\lambda > 1$.

Realisation inside ℓ_∞

After m steps,

$$Z_m = (\ell_\infty)_\infty^{N^m}.$$

This is isometric to ℓ_∞. Partition $\mathbb{N}$ into N^m infinite sets

$$\mathbb{N} = A_1 \sqcup \cdots \sqcup A_{N^m}$$

and choose bijections $\varphi_j : \mathbb{N} \to A_j$. Then

$$J(x^{(1)}, \ldots, x^{(N^m)})(\varphi_j(k)) = x_k^{(j)}$$

defines a surjective isometry

$$J : (\ell_\infty)_\infty^{N^m} \longrightarrow \ell_\infty.$$

Hence the final example can be viewed as a closed subspace of ℓ_∞.

A Banach–Mazur estimate

Recall

$$d_{\mathrm{BM}}(X, Y) = \inf\{\|T\|\,\|T^{-1}\| : T : X \to Y \text{ is an isomorphism}\}.$$

Theorem B

Let X and Y be Banach spaces such that

1. each is isometric to a 1-complemented subspace of the other;
2. $X \cong X \oplus_\infty X$ and $Y \cong Y \oplus_\infty Y$ isometrically.

Then

$$d_{\mathrm{BM}}(X, Y) \leqslant 9 + 6\sqrt{3} = 19.392304845\ldots.$$

In particular,

$$d_{\mathrm{BM}}(L_\infty[0, 1], \ell_\infty) \leqslant 9 + 6\sqrt{3}.$$

Korpalski–Plebanek obtained the explicit bound

$$(3 + \sqrt{2})^2 = 11 + 6\sqrt{2} = 19.485281374\ldots,$$

so the improvement is approximately 0.092976529.

The optimisation

The estimate comes from an explicit one-parameter family $W_a : X \to Y$ with

$$\|W_a\|, \|W_a^{-1}\| \leqslant K(a), \qquad K(a) = \frac{a+2}{a}\sqrt{2a+1} \quad (a > 0).$$

Therefore

$$d_{\mathrm{BM}}(X, Y) \leqslant g(a) := K(a)^2 = 2a + 9 + \frac{12}{a} + \frac{4}{a^2}.$$

The minimum is obtained from

$$g'(a) = 0 \quad \Longleftrightarrow \quad a^3 - 6a - 4 = 0, \qquad a = 1 + \sqrt{3}.$$

Thus

$$\min_{a>0} g(a) = g(1+\sqrt{3}) = 9 + 6\sqrt{3} = 19.392304845\ldots.$$

Conclusion

1. The distinction between (λ^+)-injectivity and λ-injectivity is controlled by non-attainment of relative projection constants.
2. The zero-sum construction satisfies

$$\lambda(\Sigma_N(Y), Z_\infty^N) = \left(2 - \frac{2}{N}\right)\lambda(Y, Z),$$

   and preserves non-attainment.
3. Choosing N and iterating reduces every $\lambda > 2$ to the already known range $(1, 2]$.
4. The complex endpoint $\lambda = 1$ remains open; a possible route is to show that complex (1^+)-injective spaces have extreme points in their unit balls.

Thank you

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