A Loomis–Sikorski quotient for Banach lattices

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Definition (Algebra of sets)

An *Algebra of sets* is a pair $(X, \mathfrak{B})$ where X is a set and $\mathfrak{B} \subset \mathcal{P}(X)$ such that

1. $\emptyset \in \mathfrak{B}$ and $X \in \mathfrak{B}$,
2. $A \in \mathfrak{B}$ implies $X \setminus A \in \mathfrak{B}$,
3. $A, B \in \mathfrak{B}$ implies $A \cap B \in \mathfrak{B}$ and $A \cup B \in \mathfrak{B}$.

Algebras and lattices

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Definition (Lattice)

A *lattice* is a partially ordered set $(X, \leq)$ such that every pair $x, y \in X$ has a supremum $x \vee y$ and an infimum $x \wedge y$.

Algebras of sets

Remark

An algebra of sets has the following properties:

1. *It is a lattice ordered by inclusion with* $A \vee B = A \cup B$ *and* $A \wedge B = A \cap B$
2. *It has a minimum element* $(\emptyset)$ *and a maximum element* (X)
3. *It is distributive (de Morgan's laws)*
4. *Every element has a complement: for every* $A \in \mathfrak{B}$ *there exists* $B \in \mathfrak{B}$ *such that* $A \wedge B = \emptyset$ *and* $A \vee B = X$

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$$A \cup B \leftrightarrow a \vee b, \qquad A \cap B \leftrightarrow a \wedge b.$$

Representation problem

Can every Boolean algebra be represented by an algebra of sets?

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Theorem (Stone's representation theorem)

Let $\mathbb{B}$ be a Boolean algebra. Its Stone space $K_{\mathbb{B}} = \mathrm{Ult}(\mathbb{B})$ is the compact Hausdorff zero-dimensional space of ultrafilters. For $a \in \mathbb{B}$, put

$$\widehat{a} = \{U \in \mathrm{Ult}(\mathbb{B}) : a \in U\}.$$

Then $a \mapsto \widehat{a}$ is a Boolean algebra isomorphism from $\mathbb{B}$ onto $\mathrm{Clop}(K_{\mathbb{B}})$.

abstract $\mathbb{B}$ $\xrightarrow{a \mapsto \widehat{a}}$ clopen sets of $K_{\mathbb{B}}$

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The measure algebra

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for every n.Since ϕ preserves countable intersections, it follows that

$$x \in \phi\left(\left[\bigcap_n B_n\right]\right) = \emptyset.$$

Theorem (Loomis–Sikorski)

For every σ-complete Boolean algebra $\mathbb{B}$, there are a σ-algebra of sets $\mathcal{A} \subset \mathcal{P}(X)$ and a σ-ideal $\mathcal{N} \subset \mathcal{A}$ such that

$$\mathbb{B} \cong \mathcal{A}/\mathcal{N}$$

as σ-complete Boolean algebras.

Banach lattices

Definition

A *Banach lattice* is a Banach space X with a lattice order such that

$$x \leq y \implies x + z \leq y + z, \quad \alpha x \leq \alpha y, \quad |x| \leq |y| \implies \|x\| \leq \|y\|.$$

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Notation

$$X_+ = \{x \in X : x \ge 0\}, \qquad |x| = x \vee (-x).$$

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Examples

- $C(K)$, pointwise order;
- $L^p(\mu)$, $1 \le p \le \infty$, a.e. order;
- the dual X^*, with $x^* \ge 0$ if $x^*(X_+) \subset \mathbb{R}_+$.

The sequential Fatou property

For $0 \leq x_n \uparrow x$, monotonicity gives

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- X is *sequentially Fatou*, or σ-Fatou, if $0 \le x_n \uparrow x \Rightarrow \|x_n\| \uparrow \|x\|$.
- X is *Fatou* if the same assertion holds for nets.
- X is *weakly Fatou* with constant $C \ge 1$ if

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We next need a concrete ambient space of Banach lattices that plays the role of Stone's theorem and $\mathcal{P}(X)$.

Banach spaces.

The canonical isometric embedding

$$X \hookrightarrow \ell_\infty(B_{X^*}), \qquad x \mapsto \big(f(x)\big)_{f\in B_{X^*}},$$

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Banach lattices.

To obtain a *lattice* embedding, one would like to mimic the same construction using *lattice homomorphisms*

$$X \longrightarrow \mathbb{R}.$$

However, unlike the Banach space setting, there may be no non-trivial lattice homomorphisms $X \to \mathbb{R}$.

Replace the scalar field $\mathbb{R}$ by L^1 spaces:

Replace the scalar field $\mathbb{R}$ by L^1 spaces: there are enough lattice homomorphisms

$$X \longrightarrow L^1(\mu),$$

to separate points, yielding an embedding of the form

$$X \hookrightarrow \ell_\infty\big((L^1(\mu_i))_{i\in I}\big).$$

Lotz representation: the ambient space

Theorem (Lotz representation)

If

$$S = \left(\bigoplus_{\varphi \in B_{X^*}^+} X_\varphi \right)_{\ell^\infty} \cong \left(\bigoplus_{\varphi \in B_{X^*}^+} L^1(\mu_\varphi) \right)_{\ell^\infty}, \qquad Jx = (j_\varphi x)_{\varphi \in B_{X^*}^+},$$

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What changes for σ-complete Banach lattices?

Even if $x_n \downarrow 0$ in X, the infimum $\inf J(x_n)$ in S need not be zero.

The Loomis–Sikorski type quotient

Definition

Let I_σ be the σ-ideal of S generated by all

$$\inf J(x_n), \qquad x_n \downarrow 0 \text{ in } X_+.$$

With $Q : S \to S/I_\sigma$, define

$$\Theta_X := QJ : X \longrightarrow S/I_\sigma.$$

By construction

Θ_X is contractive and σ-order continuous: if $u_n \downarrow 0$, then $\inf_n \Theta_X u_n = 0$.

$$\begin{tikzcd} X \arrow[r, "J"] \arrow[dr, "\Theta_X"'] & S \arrow[d, "Q"] \\ & S/I_\sigma \end{tikzcd}$$

Important limitation

We managed to preserve countable operations at a cost:

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Definition

For a Banach lattice X, we say that X has

$$\begin{aligned} \mathsf{LS}_{\mathrm{sep}} : &\quad 0 < x \in X_+ \Rightarrow \exists \phi \in A(u_n) \text{ with } \phi(x) > 0, \\ \mathsf{LS}^c_{\mathrm{nrm}} : &\quad \|x\| \le c \sup\{x^*(x) : x^* \in A(u_n)\} \quad (c \ge 1), \\ \mathsf{LS}_{\mathrm{nrm}} : &\quad \|x\| = \sup\{x^*(x) : x^* \in A(u_n)\}. \end{aligned}$$

Each condition is required for every $x \in X_+$ and every $u_n \downarrow 0$.

Faithfulness theorem

Theorem (Faithfulness of Θ_X)

Let X be a Banach lattice. Then:

1. Θ_X *is injective if and only if* X *has* $\mathsf{LS}_{\mathrm{sep}}$.
2. Θ_X *is an isomorphic embedding if and only if* X *has* $\mathsf{LS}_{\mathrm{nrm}}^{c}$ *for some* $c \geq 1$.
3. Θ_X *is an isometry if and only if* X *has* $\mathsf{LS}_{\mathrm{nrm}}$.

Faithfulness theorem

Theorem (Sequential Fatou characterization)

For every Banach lattice X,

$$\Theta_X \text{ is an isometry} \iff X \text{ has the sequential Fatou property.}$$

Equivalently, $\mathsf{LS}_{\mathrm{nrm}} \iff \sigma$*-Fatou.*

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Remark

The quotient is isometric exactly when the original norm already treats increasing sequential suprema correctly.

Examples: isometric and not isometric

Isometric cases

If X is σ-Fatou, then Θ_X is isometric. Standard examples include

$$L^p(\mu),\ 1 \le p < \infty, \qquad C(K), \qquad \text{dual Banach lattices.}$$

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AM-space Isomorphic but not isometric

Let β be a free-ultrafilter limit on $\mathbb{N}$, let $\alpha > 1$, and set

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while $\|y_n\| = 1$ and $\|x\| = \alpha$.Thus Θ_{X_α} is not isometric; nevertheless it is an isomorphic embedding.

The intermediate level is a renorming problem

Question

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Theorem (Intermediate level)

For a Banach lattice X, the following are equivalent:

1. Θ_X *is an isomorphic embedding;*
2. X *admits an equivalent σ-Fatou lattice norm.*

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Does every σ-weakly Fatou Banach lattice admit an equivalent σ-Fatou norm?

Weakly Fatou not isomorphic (Elliott; Avilés-Taylor-Tradacete)

There exists a Banach lattice with weakly Fatou norm not equivalent to a Fatou norm

Failure of injectivity

There exists a Banach lattice together with a null sequence $x_n \downarrow 0 \subset X$ for which there is no admissible functional:

$$\phi(x_n) \to \alpha_\phi > 0$$

for every $\phi \in X_+^*$

Thank you!

Selected references

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