The super Alternative Daugavet property for Banach spaces

Joint work with M. Lõo, M. Martín, Y. Perreau, and A. Rueda Zoca

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Outline of the talk

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Notation

- X is a Banach space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$)
- X^* dual space
- S_X unit sphere, B_X closed unit ball
- $\mathbb{T}$: modulus-one scalars of $\mathbb{K}$

Outline

Daugavet property

Definition (I. K. Daugavet; 1963)

A Banach space X is said to have the Daugavet property (DP) if every rank-one operator $T : X \to X$ satisfies

$$\| \mathrm{Id} + T \| = 1 + \| T \|.$$

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Example

Banach spaces with the Daugavet property include:

- $C[0,1]$ (I. K. Daugavet; 1963) and $L_\infty[0,1]$ (A. Pelczyński; 1965)
- $L_1[0,1]$ (G. Ya. Lozanovskii; 1966)
- $\mathrm{Lip}_0(M)$ and $\mathcal{F}(M)$ iff M is a length metric space (Y. Ivakhno, V. Kadets, D. Werner; 2007 + L. García-Lirola, A. Procházka, A. Rueda Zoca; 2018)
- Gurariy space (T. Abrahamsen, V. Lima, O. Nygaard; 2014)

Alternative Daugavet property

Definition (M. Martín and T. Oikhberg; 2004)

A Banach space X is said to have the Alternative Daugavet property (ADP) if every rank-one operator $T : X \to X$ satisfies

$$\max_{\theta \in \mathbb{T}} \| \mathrm{Id} + \theta T\| = 1 + \|T\|.$$

Alternative Daugavet property

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Example

Banach spaces with the ADP include:

- spaces with the Daugavet property (by definition);
- spaces with numerical index 1 (M. Martín and T. Oikhberg; 2004);
- c_0, ℓ_1, and ℓ_1^n, ℓ_∞^n, where $n \geq 2$.

Daugavet property in geometric terms

Recall that a slice of a bounded subset B of X is a set of the form

$$S(B, x^*, \alpha) := \{x \in B\colon\ \operatorname{Re} x^*(x) > \sup_{b \in B} \operatorname{Re} x^*(b) - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$.

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Lemma (R. V. Shvydkoy; 2000)

Let X be a Banach space. TFAE:

(i) *X has the Daugavet property;*

(ii) *For every $x \in S_X$ and every slice S of B_X, $\sup_{y \in S} \|x + y\| = 2$;*

(iii) *For every $x \in S_X$ and every nonempty relatively weakly open subset W of B_X, $\sup_{y \in W} \|x + y\| = 2$;*

(iv) *For every $x \in S_X$ and every convex combination C of slices of B_X, $\sup_{y \in C} \|x + y\| = 2$;*

ADP in geometric terms

Proposition (M. Martín and T. Oikhberg; 2004)

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The natural question

1. What happens if we replace slices in (ii) by relatively weakly open subsets?
2. What happens if we replace slices in (ii) by convex combinations of slices?

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Answer to Question 2: one recovers the Daugavet property.

Outline

The super Alternative Daugavet property

Definition

A Banach space X has the super Alternative Daugavet property, (super ADP), if for every $x \in S_X$ and every non-empty relatively weakly open set $W \subset B_X$ with $W \cap S_X \neq \varnothing$,

$$\sup_{y \in W} \max_{\theta \in \mathbb{T}} \|x + \theta y\| = 2.$$

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$$\sup_{y \in W} \max_{\theta \in \mathbb{T}} \|x + \theta y\| = 2.$$

- In infinite-dimensional spaces the condition $W \cap S_X \neq \varnothing$ is redundant.
- Shvydkoy's weak-open characterization of the DP gives immediately

$$\text{DP} \Longrightarrow \text{super ADP}.$$

- Since slices are relatively weakly open,

$$\text{super ADP} \Longrightarrow \text{ADP}.$$

Why ADP does not imply super ADP

Proposition

The one-dimensional space $\mathbb{K}$ is the only finite-dimensional Banach space with the super ADP. Moreover, $\mathbb{K}$ is the only super ADP space with the Kadec property.

Recall that X has the Kadec property if the identity map $\mathrm{Id}\colon (B_X, w) \to (B_X, \|\cdot\|)$ is continuous on S_X.

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Example

- Finite-dimensional spaces with the ADP, but without the super ADP, are ℓ_1^n and ℓ_∞^n for $n \geq 2$;
- Infinite-dimensional spaces with the ADP, but without the super ADP, include $\ell_1(\Gamma)$ whenever Γ is infinite.

Why super ADP does not imply DP

Theorem

If $(X, \|\cdot\|)$ has the DP, then X can be equivalently renormed so that it has the super ADP but fails the DP.

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If $(X, \|\cdot\|)$ has the DP, then X can be equivalently renormed so that it has the super ADP but fails the DP.

Fix $x_0 \in S_X$ and $0 < \varepsilon < 1$. Define an equivalent norm $|||\cdot|||$ by

$$B_{(X,|||\cdot|||)} = \operatorname{conv}\left(\mathbb{T}x_0 \cup (1-\varepsilon)B_{(X,\|\cdot\|)}\right).$$

Then, for $f \in X^*$,

$$|||f||| = \max\{(1-\varepsilon)\|f\|, |f(x_0)|\}.$$

- x_0 becomes a denting point of the new unit ball, so the renormed space fails the DP.
- The original DP supplies weakly convergent nets witnessing the super ADP for the new norm.

CPCP spaces with $\dim X \geq 2$ do not have the super ADP

Theorem

Let X be a Banach space with $\dim X \geq 2$. If X has the CPCP, then X does not have the super ADP. Consequently, spaces with the PCP or the RNP do not have the super ADP.

CPCP spaces with $\dim X \geq 2$ do not have the super ADP

Theorem

Let X be a Banach space with $\dim X \geq 2$. If X has the CPCP, then X does not have the super ADP. Consequently, spaces with the PCP or the RNP do not have the super ADP.

Idea of the proof

1. By separable determination, one may reduce to the separable case.
2. CPCP gives weak-to-norm continuity points of B_X.
3. Choose a smooth point $x \in S_X$, with support functional f, and a continuity point $y \in S_X$ with $|f(y)| < 1$. Then
$$\max_{\theta \in \mathbb{T}} \|x + \theta y\| < 2.$$
4. This strict inequality persists on a small weak neighbourhood of y, contradicting the super ADP.

Inf.-dim. super ADP spaces are not Asplund and contain ℓ_1

Recall that the norm of a Banach space X is said to be ρ-rough for $0 < \rho \leq 2$ if

$$\limsup_{\|h\|\to 0} \frac{\|x+h\| + \|x-h\| - 2\|x\|}{\|h\|} \geq \rho$$

for all $x \in X$.

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Theorem

If X is infinite-dimensional and has the super ADP, then X is 1-rough. In particular, X is not Asplund.

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If X is infinite-dimensional and has the super ADP, then X is 1-rough. In particular, X is not Asplund.

Theorem (M. Lõo and Y. Perreau; arXiv:2603)

If X is infinite-dimensional and has the super ADP, then X contains a copy of ℓ_1.

DP vs super ADP vs ADP

	DP	super ADP	ADP
Finite-dimensional	Never	Only $\mathbb{K}$	**compatible**: e.g. ℓ_1^n, ℓ_∞^n
CPCP, PCP, RNP	Fails these properties	Fails CPCP (dim. ≥ 2)	**compatible**: ℓ_1
Strong regularity	Never	Inf.-dim. case is **open**	**compatible**: ℓ_1
Roughness	2-rough	Inf.-dim. is 1-rough; 2-rough is **open**	Not forced
Asplundness	Not Asplund	Not Asplund in inf.-dim. case	**compatible**: ℓ_1^n and ℓ_∞^n
Contains ℓ_1	Yes	Yes, except for $\mathbb{K}$	No, c_0
Unconditional basis	Cannot even embed into a space with unconditional basis	Existence of an inf.-dim. example with unconditional basis is **open**	**compatible**: c_0 and ℓ_1
From X^* to X	Always	No, $X = L_1[0,1] \oplus_\infty \mathbb{K}$	Always

Outline

Pointwise versions of the Daugavet property

Definition (T. A. Abrahamsen, R. Haller, V. Lima, K. Pirk; 2020 + M. Martín, Y. Perreau, A. Rueda Zoca; 2024)

Let X be a Banach space. An element $x \in S_X$ is a

- Daugavet point if $\sup_{y \in S} \|x + y\| = 2$ for every slice S of B_X;
- super Daugavet point if $\sup_{y \in W} \|x + y\| = 2$ for every nonempty relatively weakly open subset W of B_X;
- ccs Daugavet point if $\sup_{y \in C} \|x + y\| = 2$ for every convex combination C of slices of B_X;

X has the DP $\iff$ every $x \in S_X$ is a Daugavet point;
equivalently, every $x \in S_X$ is a super Daugavet point;
equivalently, every $x \in S_X$ is a ccs Daugavet point.

Pointwise versions of the ADP and super ADP

Definition

Let X be a Banach space. An element $x \in S_X$ is

- an alternative Daugavet point if $\sup_{y \in S} \max_{\theta \in \mathbb{T}} \|x + \theta y\| = 2$ for every slice S of B_X;
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- X has the ADP $\iff$ every $x \in S_X$ is an AD point.
- X has the super ADP $\iff$ every $x \in S_X$ is a super AD point.

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- X has the ADP $\iff$ every $x \in S_X$ is an AD point.
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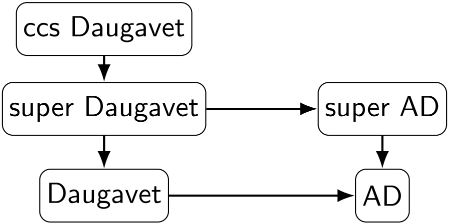
Remark

The hypothetical "ccs AD point" notion collapses to ccs Daugavet point.

Relations between pointwise notions



Relations between pointwise notions

$$
\begin{tikzcd}
\text{ccs Daugavet} \arrow[d] & \\
\text{super Daugavet} \arrow[r] \arrow[d] & \text{super AD} \arrow[d] \\
\text{Daugavet} \arrow[r] & \text{AD}
\end{tikzcd}
$$

Example

- c_0 has the ADP, but c_0 has no super AD point.
- ℓ_1 has the ADP, but ℓ_1 has no Daugavet points.
- $C[0,1] \oplus_1 \mathbb{K}$ has the super ADP, but $(0,1)$ is a denting point, hence cannot even be a Daugavet point.

Outline

Open questions

J. Langemets, M. Lõo, M. Martín, Y. Perreau and A. Rueda Zoca,
The super Alternative Daugavet property for Banach spaces,
Proc. Edinb. Math. Soc. **69** (2026), 471–499.

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Problem

Does there exist an infinite-dimensional super ADP space, which is strongly regular?

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Does there exist an infinite-dimensional super ADP space without Daugavet points?

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Problem

Does every infinite-dimensional super ADP space have a 2-rough norm?