On problems related to the slicely countably determined property

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Notation

- X real Banach space, X^* dual space
- S_X unit sphere, B_X closed unit ball
- $\mathrm{conv}(\cdot)$ convex hull, $\overline{\mathrm{conv}}(\cdot)$ closed convex hull

Let $A \subset X$ be bounded. A slice of A is a (nonempty and relatively weakly open) subset of A the form

$$S(A, x^*, \alpha) := \{x \in A : x^*(x) > \sup x^*(A) - \alpha\} \quad (x^* \in X^*, \alpha > 0)$$

SCD sets and spaces

Let X be a Banach space and $A \subset X$ be bounded.

Definition (Avilés, Kadets, Martín, Merí, Shepelska (2010))

A sequence $\{V_n \colon n \in \mathbb{N}\}$ of subsets of A is determining for A, if one of the following equivalent conditions hold:

- if $x_n \in V_n$ for every n, then $A \subset \overline{\mathrm{conv}}(\{x_n \colon n \in \mathbb{N}\})$;
- if for every slice S of A, there is a V_m such that $V_m \subset S$.

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Definition

The set A is said to be slicely countably determined (an SCD set in short), if there exists a determining sequence of slices of A.

Note that SCD sets are necessarily separable. Hence, we stick to separable sets and Banach spaces.

Examples of SCD sets

Given a closed, bounded and convex $A \subset X$, a point $x \in A$ is called a **denting point** of A if for every $\varepsilon > 0$ there exists a slice S of A such that $x \in S$ and $\operatorname{diam}(S) < \varepsilon$. We denote the set of denting points of A as $\operatorname{dent}(A)$. A closed, bounded and convex set A is called **dentable**, if $A = \overline{\operatorname{conv}}(\operatorname{dent}(A))$ (i.e. every slice contains a denting point).

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Example (Small slices)

If A is dentable, then A is SCD. For instance, this is true for closed bounded convex subsets of ℓ_p, $1 \leq p < \infty$.

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Example (Reasonable size dual)

If A contains no ℓ_1-sequences (in particular, if X^* is separable), then A is SCD. This is true for closed bounded convex subsets of c_0.

Therefore, even sets with big slices (for instance, every slice of B_{c_0} has diameter 2) can be SCD.

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- Spaces with RNP (all closed bounded convex subsets are dentable).
- Spaces having a separable dual (equivalently, dual with the RNP). More generally, spaces not containing a copy of ℓ_1.
- All separable reflexive Banach spaces are SCD.

The opposite of SCD: Daugavet property

Definition (Kadets, Shvidkoy, Sirotkin, Werner (1997))

Banach space X has the Daugavet property, if for every $x \in S_X$, slice S of B_X and $\varepsilon > 0$, there exists $y \in S$ such that $\|x - y\| > 2 - \varepsilon$.

Examples of spaces with the Daugavet property are $C[0,1]$ and $L_1[0,1]$.

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Problem (Avilés, Kadets, Martín, Merí, Shepelska (2010))

- Does every separable Banach space that is not SCD possess the Daugavet property in some equivalent norm?

Where to find a counterexample?

It is known that spaces with the Daugavet property do not embed into spaces with unconditional basis (Kadets, Shvidkoy, Sirotkin, Werner (1997)). Thus, it would be enough to produce a space which has an unconditional basis and is not SCD.

Banach space $X_{\mathcal{T}}$ generated by the infinite binary tree

Denote $\mathcal{T} := \bigcup_{n=0}^{\infty}\{0,1\}^n$, that is, all finite sequences of zeros and ones. Let $\preceq$ be the natural order on $\mathcal{T}$ which produces an infinite binary tree.

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- a chain of $\mathcal{T}$ is any well-ordered subset of $\mathcal{T}$;
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Let $\mathcal{B}$ be the collection of all chains of $\mathcal{T}$. This forms an adequate family of subsets (in the sense of Talagrand (1979)). For $x \in c_{00}(\mathcal{T})$, write $x = \sum_{s\in T_0} x(s)e_s$, where $T_0 \subset \mathcal{T}$ is finite. We define a norm on $\mathcal{T}$ as

$$\Big|\Big|\sum_{s\in T_0} x(s)e_s\Big|\Big| = \sup_{B\in\mathcal{B}}\sum_{s\in B}|x(s)|.$$

We consider the completion of $c_{00}(\mathcal{T})$ with this norm, denoted as $X_{\mathcal{T}}$.

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- Given any branch $\beta = (\beta(n))_n$ of $\mathcal{T}$, the subspace $\overline{\operatorname{span}}\{e_{\beta(n)} \colon n \in \mathbb{N}\}$ is isometric to ℓ_1.

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- Given any antichain $\alpha = (\alpha(n))_n$ of $\mathcal{T}$, the subspace $\overline{\text{span}}\{e_{\alpha(n)} \colon n \in \mathbb{N}\}$ is isometric to c_0. In particular, $e_{\alpha(n)} \to 0$ weakly.
- Every Banach space with a 1-unconditional basis embeds into $X_{\mathcal{T}}$ isometrically (Bang, Odell (1989)). Thus $X_{\mathcal{T}}$ is SCD if and only if every Banach space with a 1-unconditional basis is SCD.

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Theorem (Kadets, Martín, Merí, Werner (2011))

Let X be a Banach space with 1-unconditional basis. Then B_X is an SCD set.

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Proposition

$x \in \mathrm{dent}(B_{X_{\mathcal{T}}})$ *if and only if there exists a finite maximal antichain* α *of* T *such that* $|x(s)| = 1$ *for every* $s \in \alpha$.

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Theorem

The subset $B^{+}_{X_{\mathcal{T}}} = \{x \in B_{X_{\mathcal{T}}} \colon x(s) \geq 0 \ \forall s \in T\}$ is not an SCD set. Therefore, $X_{\mathcal{T}}$ is not an SCD space.

Countable π-basis for the relative weak topology

Let $A \subset X$ be convex and bounded.

Definition

We say that A has a countable π-basis (pseudobasis) for (A, w), if there exists $\{W_n \subset A \colon n \in \mathbb{N}\}$ of relatively weakly open subsets such that for every relatively weakly open subset $W \subset A$ there is $m \in \mathbb{N}$ such that $W_m \subset W$.

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Proposition

If A has a countable π-basis for (A, w), then it is an SCD set.

Problem (Avilés, Kadets, Martín, Merí, Shepelska (2010))

Is it true that an SCD set A always admits a countable π-basis for (A, w)? One way to approach it: is it true that (B_X, w) always admits a countable π-basis, given X with 1-unconditional basis?

Countable π-basis for the relative weak topology

Recall that $x \in S_X$ is a **point of continuity** of B_X, if for every $\varepsilon > 0$ there is a relatively weakly open subset W of B_X such that $x \in W$ and $\operatorname{diam} W < \varepsilon$. We denote $PC(B_X)$ for the set of points of continuity of B_X.

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Proposition

Assume X is a separable Banach space. If $PC(B_X)$ is weakly dense in B_X (every W contains a point of continuity), then (B_X, w) admits a countable π-basis.

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Proposition

A point $x \in S_{X_{\mathcal{T}}}$ is a point of continuity of $B_{X_{\mathcal{T}}}$ if and only if $\sum_{s\in\beta} |x_s| = 1$ for every branch β of $\mathcal{T}$.

Theorem

In $X_{\mathcal{T}}$ we have $PC(B_{X_{\mathcal{T}}})$ weakly dense in $B_{X_{\mathcal{T}}}$. Hence, $(B_{X_{\mathcal{T}}}, w)$ admits a countable π-basis.

Thank you!