A small $\mathcal{C}(K)$ space without nice renormings

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based on a joint work with D. Sobota and L. Zdomskyy

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09/07/2026

Summer School: Topics in Banach Space Theory

CIEM Castro Urdiales, Spain

Strictly convex renormings at density ω_1 in models of $\neg CH$?

Asked 2 months ago Modified 2 months ago Viewed 265 times

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I have a question about the consistency strength of a renorming problem that, as far as I understand, goes back to Avilés.

Avilés asked whether there exists, in ZFC, a Banach space of density ω_1 that admits no equivalent strictly convex norm.

There are of course classical Banach spaces with no equivalent strictly convex renorming, such as ℓ_∞/c_0, and also $\ell_\infty(\omega_1)$. However, these do not settle the density-ω_1 problem: ℓ_∞/c_0 has density $\mathfrak{c}$, while $\ell_\infty(\omega_1)$ is also too large in general from the point of view of Avilés's question.

Under CH, one can use $\ell_\infty^c(\omega_1)$, whose density is then ω_1, so the interesting case is really when CH fails.

My question is:

> Is it known that in some model of $\neg CH$ there exists a Banach space of density ω_1 with no equivalent strictly convex renorming?

More specifically, is such an example known under any of the standard forcing axioms (for instance $MA + \neg CH$, PFA, or MM), or in some tailor-made forcing extension?

I would also be very interested in partial results in either direction. For example:

$$\text{(a) consistency results producing such a space under } \neg CH,$$

or

$$\text{(b) structural consequences of forcing axioms that rule out the usual candidates.}$$

Any references or pointers would be very welcome.

fa.functional-analysis set-theory banach-spaces forcing

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asked Apr 3 at 14:03
Tomasz Kania

1 Answer

Sorted by: Highest score (default)

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Haydon, Jayne, Namioka, and Rogers proved that if S is a Souslin line, then $C_0(S)$ does not have a strictly convex renorming (Example 3 in this paper) but it has density ω_1 as this is the weight of S. Such a line may exist under the negation of the Continuum Hypothesis.

So the real question is whether there is a ZFC example.

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answered Apr 24 at 7:47

Tomasz Kania

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A SMALL BANACH SPACE $C(K)$ WITHOUT NICE RENORMINGS

TODOR MANEV, DAMIAN SOBOTA, AND LYUBOMYR ZDOMSKYY

Some definitions

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- *midpoint locally uniformly rotund* ($MLUR$) [resp. *weakly midpoint locally uniformly rotund* ($wMLUR$)], if whenever for $x \in S_X$ and $\langle y_n \in X \colon\ n \in \omega \rangle$ we have $\|x \pm y_n\| \to 1$, then $y_n \to 0$ [resp., $y_n \xrightarrow{w} 0$],

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- *Kadets–Klee* (KK) [resp. *sequentially Kadets–Klee* (SKK)], if the weak topology and the norm topology coincide on S_X [resp., if weakly convergent sequences in S_X are norm convergent].

Some implications

$$
\begin{tikzcd}
\mathsf{KK} \arrow[r, Rightarrow] & \mathsf{SKK} & \\
\mathsf{LUR} \arrow[u, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathsf{MLUR} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathsf{R} \\
\mathsf{wLUR} \arrow[r, Rightarrow] & \mathsf{wMLUR} \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

The question

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Is there a Banach space of density $< \mathfrak{c}$ that fails to have a renorming with any (all) of the above properties?

Some answers

Theorem (Haydon, Jayne, Namioka, Rogers ('00))

Let S be the two-point compactification of a Suslin line. Then $C(S)$ admits no strictly convex renorming.

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$$\begin{array}{ccc} \mathrm{KK} & \Longrightarrow & \mathrm{top}\big(w \cup \{A_n \colon n \in \omega\}\big) \supseteq \tau_{\|\cdot\|} \\ & & \Big\Downarrow \\ & & \text{Every } \|\cdot\|\text{-discrete subset is } w\text{-}\sigma\text{-scattered} \end{array}$$

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Example

ℓ_∞ fails all properties except for $\langle \mathrm{R} \rangle$. ℓ_∞ / c_0 fails all properties.

Remark

$\ell_\infty = \mathcal{C}(St(\mathcal{P}(\omega)))$. $\ell_\infty / c_0 = \mathcal{C}\left(St(\mathcal{P}(\omega)/Fin)\right)$

Some definitions

Definition

A Boolean algebra $\mathcal{A}$ has the *Interpolation Property* ((*I*)) if for any two disjoint countable antichains $\langle A_n\colon\ n \in \omega\rangle$ and $\langle B_n\colon\ n \in \omega\rangle$ in $\mathcal{A}$ there is $A \in \mathcal{A}$ such that $A_n \leq A$ and $A \wedge B_n = 0_{\mathcal{A}}$ for all $n \in \omega$.

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Definition

A Boolean algebra $\mathcal{A}$ has the *Countable Non-Completeness Property* (*(CNC)*) if for any countable antichain $\langle A_n \colon\ n \in \omega \rangle$ in $\mathcal{A}$ there is $A \in \mathcal{A} \setminus \{0_{\mathcal{A}}\}$ such that $A \wedge A_n = 0_{\mathcal{A}}$ for all $n \in \omega$.

Main preliminary results

Theorem (Alexandrov, Babev ('88))

If $\mathcal{A}$ has (I), then $\mathcal{C}\left(St(\mathcal{A})\right)$ has no equivalent weakly midpoint-locally uniformly rotund norm.

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If $\mathcal{A}$ has (I) and (CNC), then $\mathcal{C}(St(\mathcal{A}))$ has no equivalent stricltly convex norm.

Idea of the proof

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- Certainly $f_n \overset{\|\cdot\|}{\not\to} f$, but maybe $f_n \xrightarrow{w} f$.
- If f is obtained by some countable join, then use (CNC) to find two extensions with the same norm.

The problem

Observation

Let $\mathcal{A}$ be an infinite Boolean algebra with (I). Then $|\mathcal{A}| \geq \mathfrak{c}$.

The main inspiration

C. Brech, *On the density of Banach spaces $C(K)$ with the Grothendieck property*, Proc. Amer. Math. Soc. 134 (2006), no 12, 3653–3663.

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No idea.

The result

Theorem

Let $\mathcal{A} \in V$ be a Boolean algebra which has property (I) in V. Let G be an $\mathbb{S}_\kappa$-generic filter over V. Then, in $V[G]$ the Banach space $\mathcal{C}(St(\mathcal{A}))$ admits no equivalent wMLUR or SKK norm

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Theorem

Let $\mathcal{A} \in V$ be a Boolean algebra which has properties (I) and (CNC) in V. Let G be an $\mathbb{S}_\kappa$-generic filter over V. Then, in $V[G]$ the Banach space $\mathcal{C}(St(\mathcal{A}))$ admits no equivalent strictly convex norm.

Thank you for the attention!