Weak*-convergence of sequences in dual Banach spaces

based on joint work with José Rodríguez

Summer School 2026: Topics in Banach Space Theory
Castro Urdiales

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Therefore the Grothendieck property is not inherited by arbitrary subspaces.

You can learn more about Grothendieck spaces in
M. González and T. Kania, *Grothendieck spaces: the landscape and perspectives*, Japanese Journal of Mathematics 16 (2021), 247–313.

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Actually, Problem 22 can be answered with $\mathfrak{s}$ in place of $\mathfrak{p}$.

Weak*-extensibility

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A subspace $Y \subseteq X$ is *w^*-extensible in X* if, for every w^*-convergent sequence $(y_n^*)_{n\in\mathbb{N}}$ in Y^*, there are

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The point is that this is a sequential lifting property for the restriction map

$$X^* \longrightarrow Y^*.$$

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Since X is Grothendieck, (x_k^*) is weakly convergent in X^*.
Restricting to Y, the subsequence $(y_{n_k}^*)$ is weakly convergent in Y^*.
This sequential criterion implies that Y is Grothendieck. $\square$

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What can we say about $\overline{\{z_n^* : n \in \mathbb{N}\}}^{w^*}$?

The kernel of the restriction map

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Then (K, w^*) is compact and homeomorphic to $(B_{(X/Y)^*}, w^*)$.

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Hence

L is a **countable discrete extension** of K,

i.e. L is a compact space obtained by adding to K a countable discrete set.

The proof becomes topological

The sequence (z_n^*) is a sequence in the CDE

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If L is sequentially compact, then (z_n^*) has a w^*-convergent subsequence.

So the only remaining task is to guarantee sequential compactness of L from the quotient X/Y.

Our main question question now is:

when is a CDE of a sequentially compact compactum again sequentially compact?

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So one needs a reason for the sequential compactness of K that survives after adding countably many isolated points.

A hereditary G_δ condition

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This property is inherited by closed subspaces.

It is stable under countably discrete extensions.

The small-weight condition

Let $\mathfrak{s}$ be the splitting number.

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For the present purpose one may use the following topological characterization:

$$\begin{aligned}\mathfrak{s} = \min\{\, \kappa : \ & \text{there exists a compact Hausdorff space } K \\ & \text{with } \operatorname{weight}(K) = \kappa \\ & \text{which is not sequentially compact}\,\}.\end{aligned}$$

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Let L be a CDE of an infinite compact space K. Then

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Therefore, if $\mathrm{weight}(K) < \mathfrak{s}$, then L is sequentially compact.

Summarizing the main results

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Lemma

Let L be a countably discrete extension of a closed set K.

If either

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Main Theorem

Let $Y \subseteq X$ be a closed subspace. Suppose that X/Y satisfies one of the following conditions:

1. every non-empty w^*-closed subset of $B_{(X/Y)^*}$ has a G_δ-point;
2. $\mathrm{dens}(X/Y) < \mathfrak{s}$.

Then Y is w^*-extensible in X.

Applications

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Let $Y \subseteq X$ be a closed subspace.
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The proof uses that these classes give dual balls whose non-empty w^*-closed subsets have G_δ-points.

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Corollary

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Since $\mathfrak{p} \leq \mathfrak{s}$, this answers Problem 22.

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In particular, there are Banach spaces X and separable subspaces $Y \subseteq X$ such that

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while

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Proposition

Let $Y \subseteq X$ be w^*-extensible in X. If both

$$(B_{Y^*}, w^*) \quad \text{and} \quad (B_{(X/Y)^*}, w^*)$$

are sequentially compact, then (B_{X^*}, w^*) is sequentially compact.

Proof idea

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Since (B_{Y^*}, w^*) is sequentially compact, pass to a subsequence such that

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Passing to a further subsequence if necessary,

$$x_n^* = u_n^* + v_n^*$$

converges in the weak* topology of X^*.

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This generalizes a result of Hagler and Sullivan who proved that (B_{X^*}, w^*) is sequentially compact whenever there is a subspace $Y \subseteq X$ such that

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- (B_{Y^*}, w^*) is sequentially compact;
- X/Y admits an equivalent Gâteaux smooth norm.

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It is obvious that any CDE of a compact space with countable tightness has countable tightness. It is consistent with ZFC that countable tightness implies sequential compactness.
Therefore, it is consistent that any CDE of a compact space with countable tightness is sequentially compact.

Nevertheless, Alan Dow proved in 2017 that **it is consistent that there is a Fréchet-Urysohn compact space with a CDE which is not sequentially compact.**

References

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Thank you!