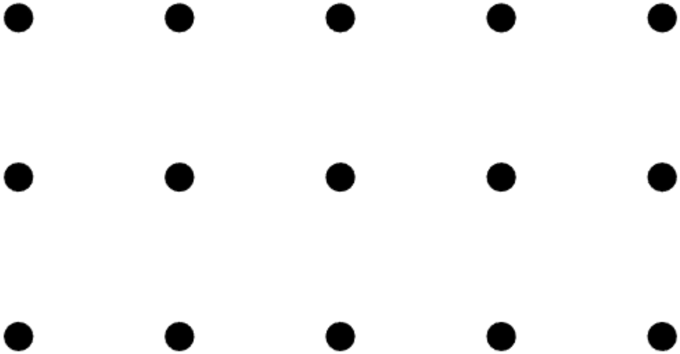
The separation modulus and the Nagata dimension

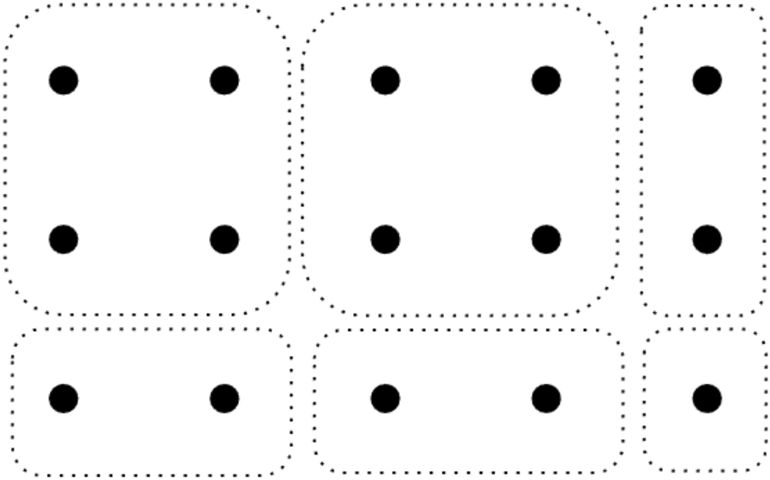
Rubén Medina
ruben.medina@unavarra.es

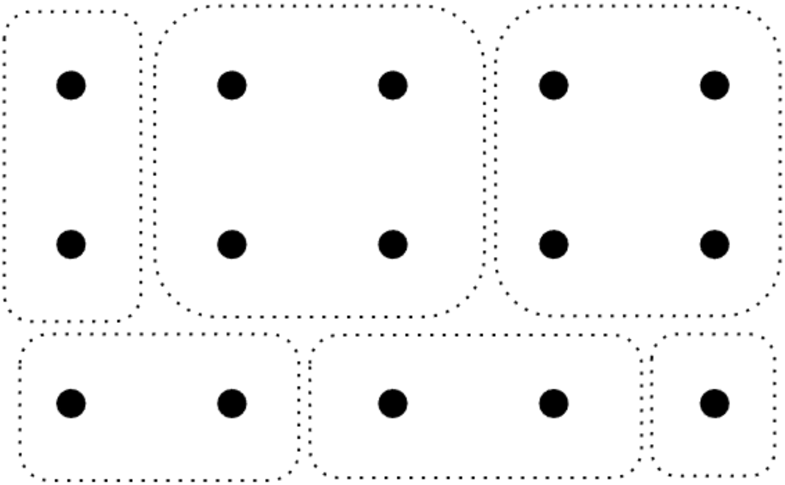
Joint work with
Ramón Aliaga

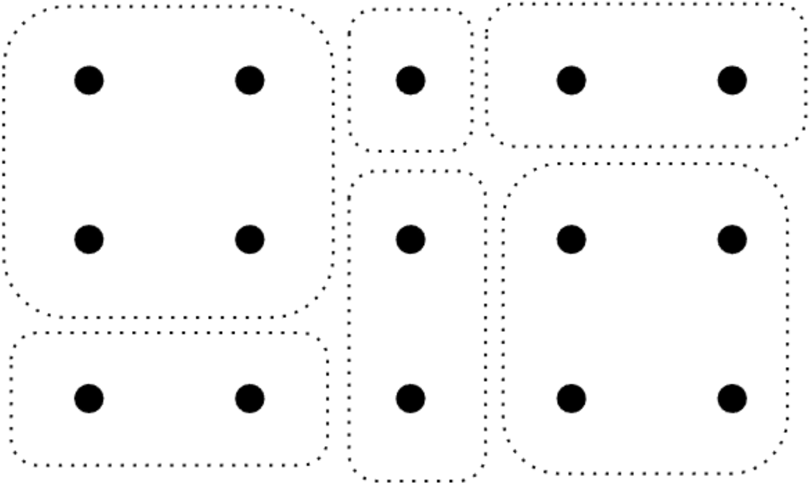
Public University of Navarre

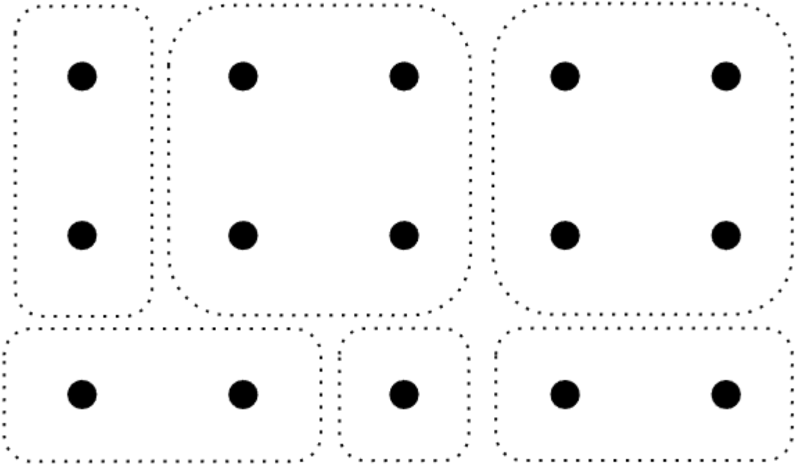
01/2026

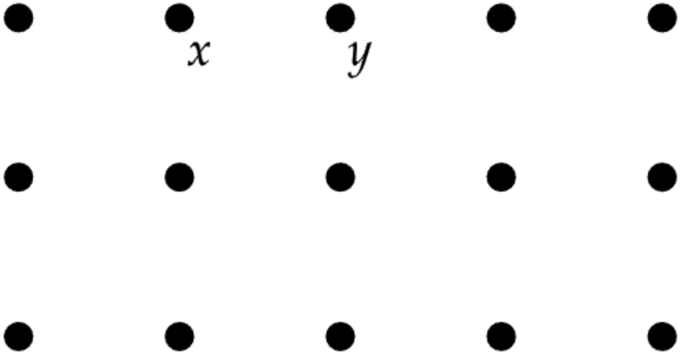


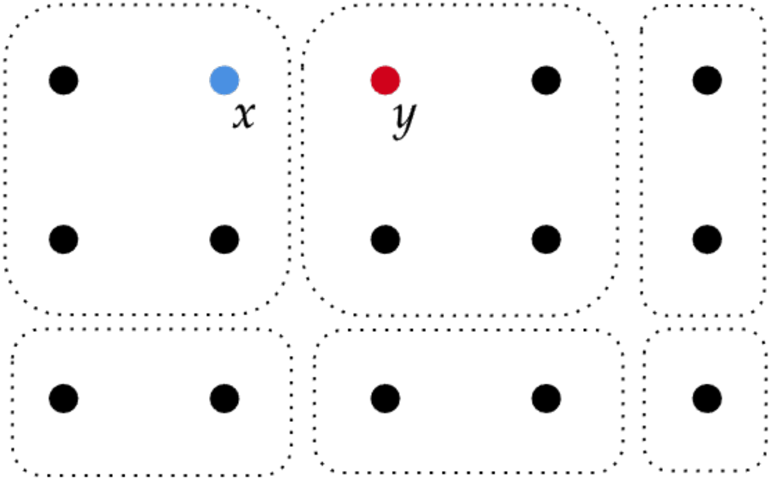


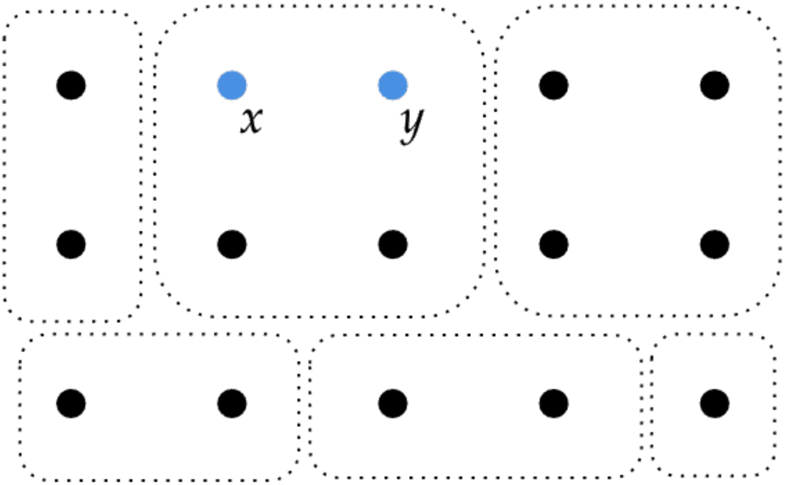




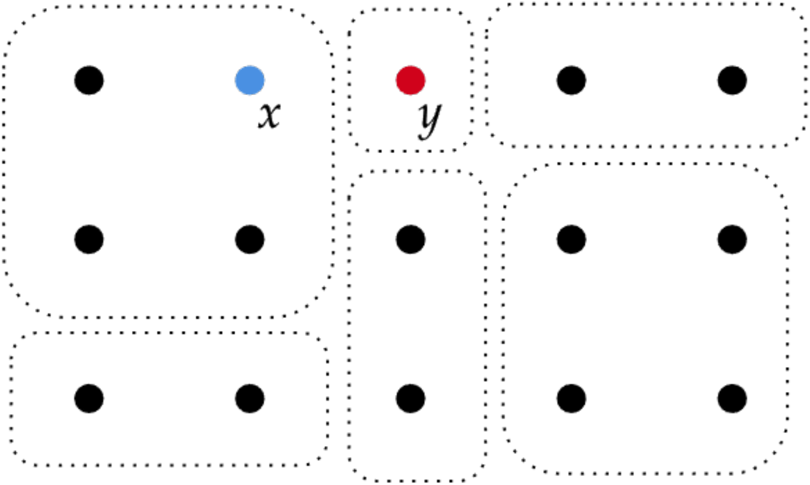


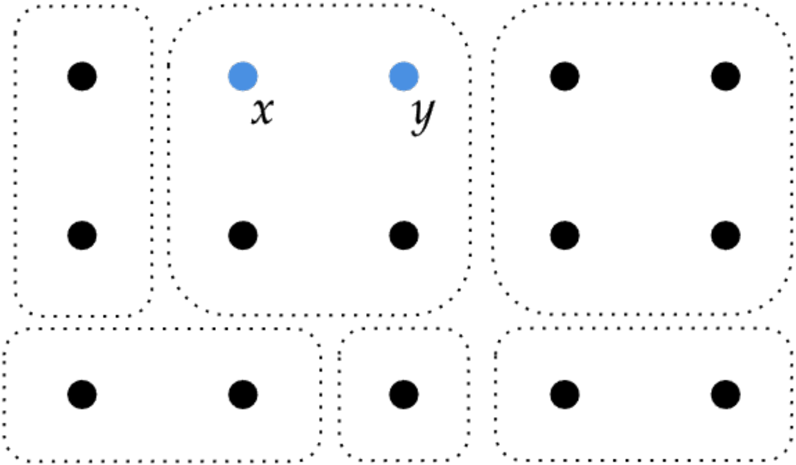

x
y


x
y



x
y


x
y


x
y

-(M, d) will be a metric space.

-(M, d) will be a metric space.
-A D-partition of M: A partition P of M where the diameter of each cluster is bounded by $D > 0$.

-(M, d) will be a metric space.
-A D-partition of M: A partition P of M where the diameter of each cluster is bounded by $D > 0$.
-If $x \in M$, $P(x)$ is the cluster of P containing x.

-(M, d) will be a metric space.
-A D-partition of M: A partition P of M where the diameter of each cluster is bounded by $D > 0$.
-If $x \in M$, $P(x)$ is the cluster of P containing x.

Definition (M finite. Y. Bartal '96 and A. Naor '04)

The separation modulus of M, $\mathrm{SEP}(M)$

-(M, d) will be a metric space.
-A D-partition of M: A partition P of M where the diameter of each cluster is bounded by $D > 0$.
-If $x \in M$, $P(x)$ is the cluster of P containing x.

Definition (M finite. Y. Bartal '96 and A. Naor '04)

The separation modulus of M, $\mathrm{SEP}(M)$, is the least constant $\lambda > 0$ such that, for every $D > 0$ there is a probability p over the D-partitions of M satisfying

$$p(P(x) \neq P(y)) \leq \frac{\lambda d(x, y)}{D}, \quad \forall x, y \in M$$

-(M, d) will be a metric space.
-A D-partition of M: A partition P of M where the diameter of each cluster is bounded by $D > 0$.
-If $x \in M$, $P(x)$ is the cluster of P containing x.

Definition (M finite. Y. Bartal '96 and A. Naor '04)

The separation modulus of M, $\mathrm{SEP}(M)$, is the least constant $\lambda > 0$ such that, for every $D > 0$ there is a probability p over the D-partitions of M satisfying

$$p(P(x) \neq P(y)) \leq \frac{\lambda d(x, y)}{D}, \quad \forall x, y \in M$$

-If M is not finite we consider

$$\mathrm{SEP}(M) = \sup_{S \subset M \text{ finite}} \mathrm{SEP}(S).$$

Applications of SEP(M)

Applications of SEP(M)

-Y. Bartal '96: Construction of stochastic embeddings into trees.

Applications of SEP(M)

-Y. Bartal '96: Construction of stochastic embeddings into trees.
-J. Lee and A. Naor '04: Extension of Lipschitz mappings.

Applications of SEP(M)

-Y. Bartal '96: Construction of stochastic embeddings into trees.
-J. Lee and A. Naor '04: Extension of Lipschitz mappings.
-A. Chang, A. Naor and K. Ren '25: Random zero sets and quasisymmetric embeddings into ℓ_2.

Applications of SEP(M)

-Y. Bartal '96: Construction of stochastic embeddings into trees.
-J. Lee and A. Naor '04: Extension of Lipschitz mappings.
-A. Chang, A. Naor and K. Ren '25: Random zero sets and quasisymmetric embeddings into ℓ_2.

Theorem (Lee-Naor-Silberman)

If M has finite Nagata dimension then $SEP(M) < \infty$.

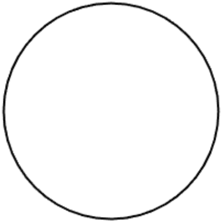
Applications of SEP(M)

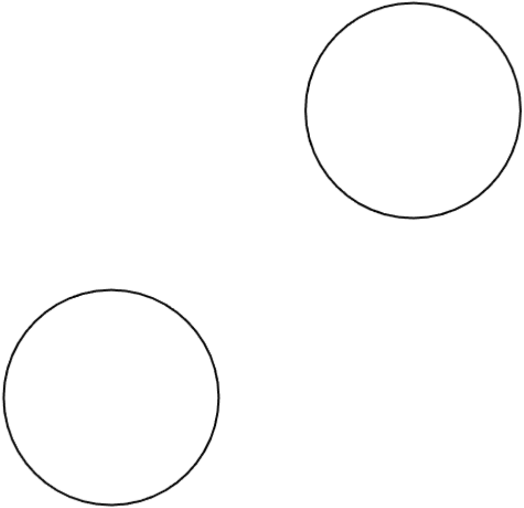
-Y. Bartal '96: Construction of stochastic embeddings into trees.
-J. Lee and A. Naor '04: Extension of Lipschitz mappings.
-A. Chang, A. Naor and K. Ren '25: Random zero sets and quasisymmetric embeddings into ℓ_2.

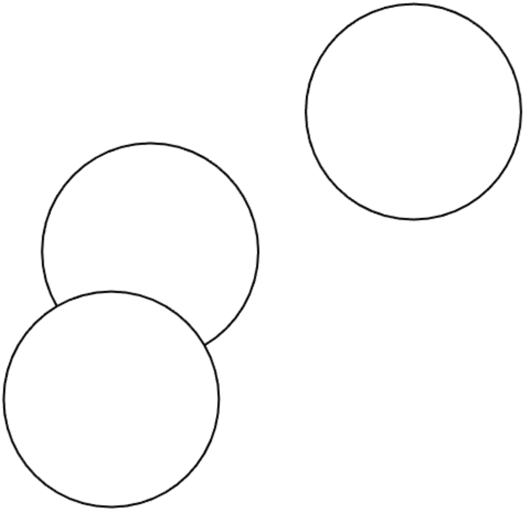
Theorem (Lee-Naor-Silberman)

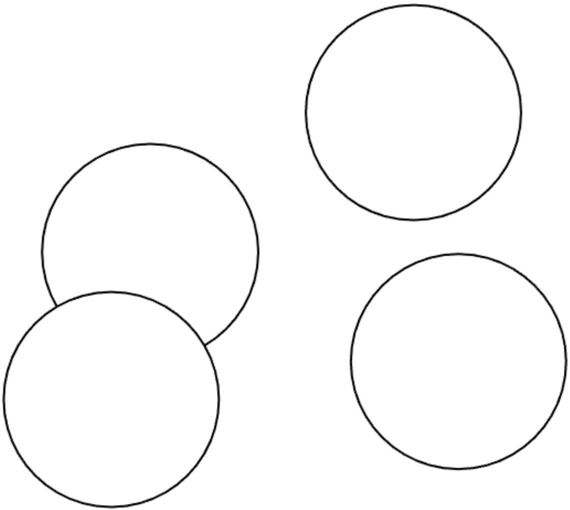
If M has finite Nagata dimension then $SEP(M) < \infty$.

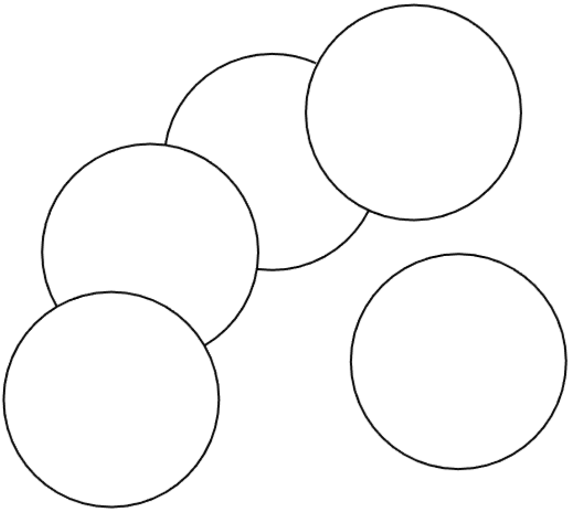
-Example: $\mathrm{SEP}(\mathbb{R}^d)$.

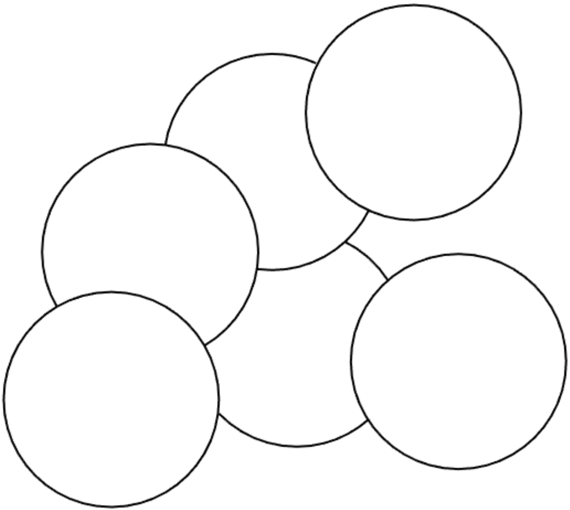


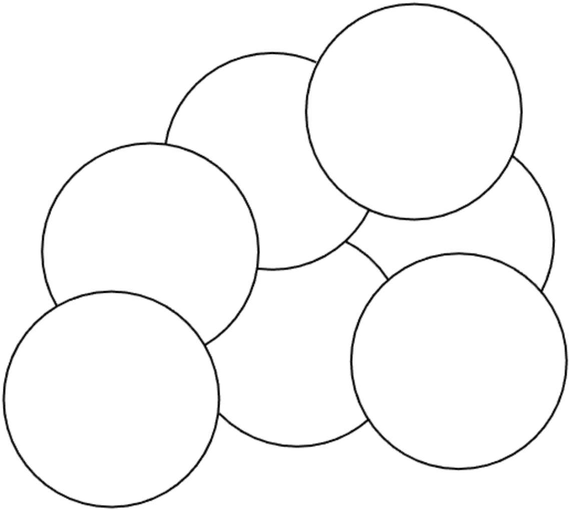


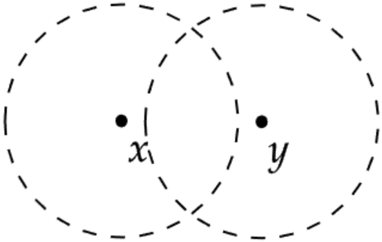


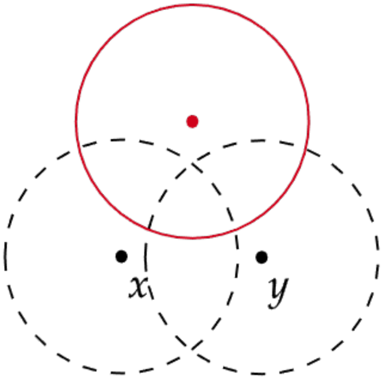




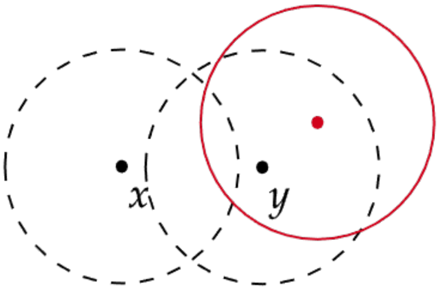




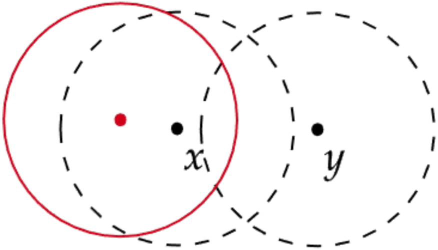

x
y



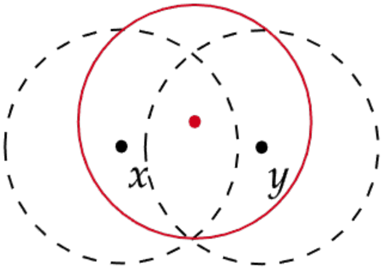
$P(x),\ P(y)?$



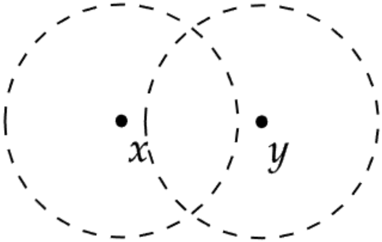
$$P(x) \neq P(y)$$



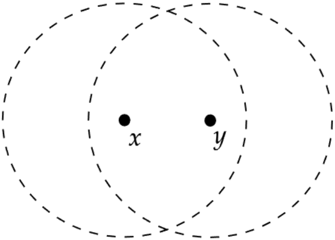
$$P(x) \neq P(y)$$

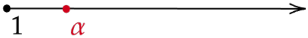


$$P(x) = P(y)$$

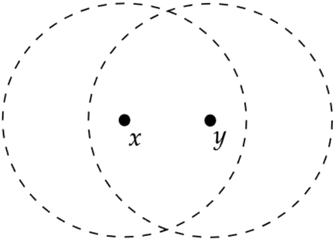


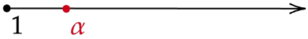
$$p(P(x) \neq P(y)) = \frac{\lambda(B(x) \Delta B(y))}{\lambda(B(x) \cup B(y))}$$


x
y



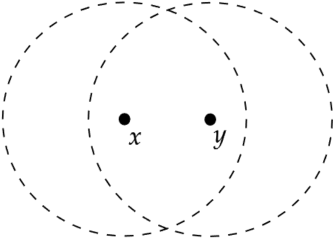
$$\rho = \log_{\alpha}(\alpha \|\cdot\|)$$

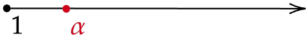




$$\rho = \log_{\alpha}(\alpha||\cdot||)$$

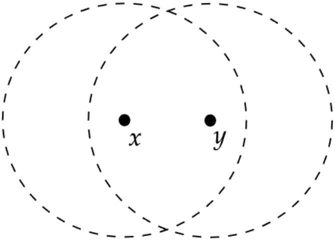
$$\rho(x,y) \leq \frac{D}{2}$$

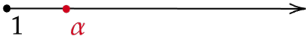




$$\rho = \log_{\alpha}(\alpha || \cdot ||)$$

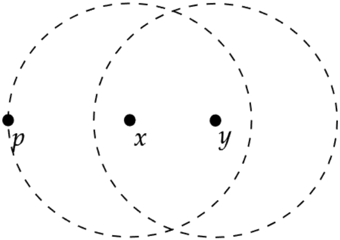
$$\rho(x,y) \leq \frac{D}{2} \implies ||x-y|| \leq \alpha^{\frac{D}{2}}$$

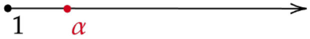




$$\rho = \log_\alpha(\alpha||\cdot||)$$

$$\rho(x,y) \leq \frac{D}{2} \implies ||x-y|| \leq \alpha^{\frac{D}{2}}$$

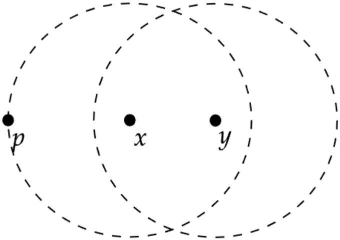


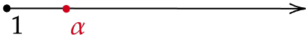


$$\rho = \log_\alpha(\alpha||\cdot||)$$

$$\rho(x,y) \le \frac{D}{2} \implies ||x-y|| \le \alpha^{\frac{D}{2}}$$

$$\rho(x,p) = D$$

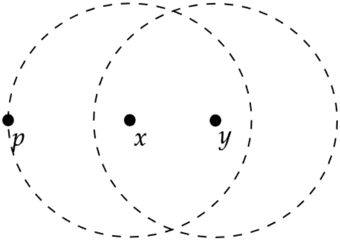


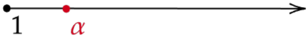


$$\rho = \log_\alpha(\alpha||\cdot||)$$

$$\rho(x,y) \le \frac{D}{2} \implies ||x-y|| \le \alpha^{\frac{D}{2}}$$

$$\rho(x,p) = D \implies ||x-p|| = \alpha^D$$

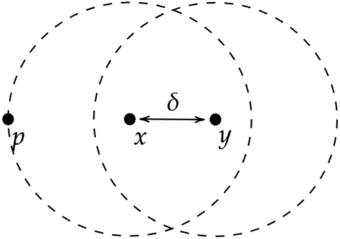


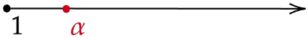


$$\rho = \log_\alpha(\alpha||\cdot||)$$

$$\rho(x,y) \leq \frac{D}{2} \implies ||x-y|| \leq \alpha^{\frac{D}{2}}$$

$$\rho(x,p) = D \implies ||x-p|| = \alpha^{D}$$

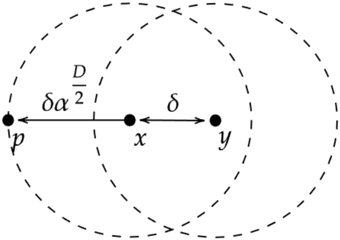


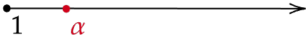


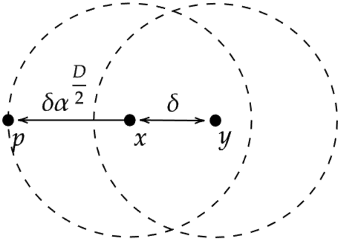
$$\rho = \log_\alpha(\alpha||\cdot||)$$

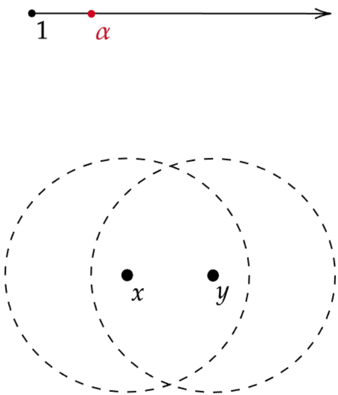
$$\rho(x,y) \le \frac{D}{2} \implies ||x-y|| \le \alpha^{\frac{D}{2}}$$

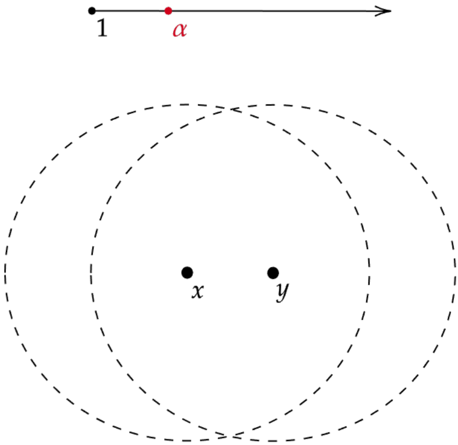
$$\rho(x,p) = D \implies ||x-p|| = \alpha^D$$

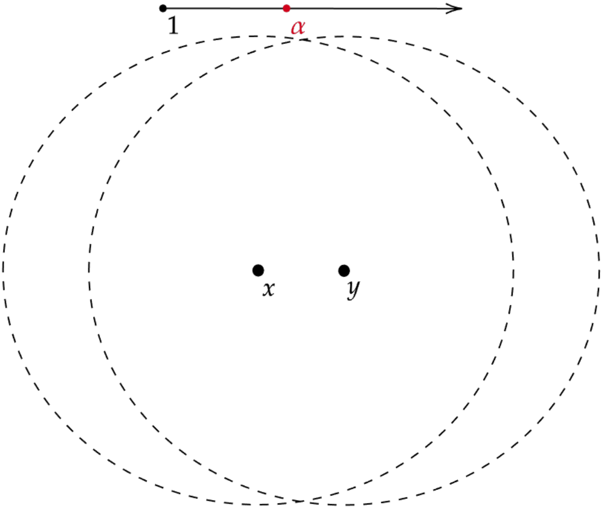



1
α


$\delta\alpha^{\frac{D}{2}}$
δ
p
x
y


1
α
x
y


1
α
x
y


1
α
x
y

Example

We denote M_d the metric space $\left(\mathbb{Z}^d, \log_{d^2}(d^2\|\cdot\|)\right)$.

Example

We denote M_d the metric space $\left(\mathbb{Z}^d, \log_{d^2}(d^2\|\cdot\|)\right)$.
We denote $\rho(x, y) = \log_{d^2}(d^2\|x - y\|)$ for every $x, y \in \mathbb{Z}^d$.

Example

We denote M_d the metric space $\left(\mathbb{Z}^d, \log_{d^2}(d^2\|\cdot\|)\right)$.
We denote $\rho(x,y) = \log_{d^2}(d^2\|x-y\|)$ for every $x, y \in \mathbb{Z}^d$.

Theorem

The Nagata dimension of M_d is d for every $d \in \mathbb{N}$.

Example

We denote M_d the metric space $\left(\mathbb{Z}^d, \log_{d^2}(d^2\|\cdot\|)\right)$.
We denote $\rho(x, y) = \log_{d^2}(d^2\|x - y\|)$ for every $x, y \in \mathbb{Z}^d$.

Theorem

The Nagata dimension of M_d is d for every $d \in \mathbb{N}$.

Theorem

$SEP(M_d) \leq 64$ for every $d \in \mathbb{N}$.

Example

We denote M_d the metric space $\left(\mathbb{Z}^d, \log_{d^2}(d^2\|\cdot\|)\right)$.
We denote $\rho(x,y) = \log_{d^2}(d^2\|x-y\|)$ for every $x, y \in \mathbb{Z}^d$.

Theorem

The Nagata dimension of M_d is d for every $d \in \mathbb{N}$.

Theorem

$SEP(M_d) \leq 64$ for every $d \in \mathbb{N}$.

Theorem

There is a universal $C > 0$ such that $\mathcal{F}(M_d)$ is isomorphic to ℓ_1 with distortion $\leq C$ for every $d \in \mathbb{N}$.

Conclusions

Conclusions

The Nagata dimension is not equivalent to the Separation modulus.

Conclusions

The Nagata dimension is not equivalent to the Separation modulus.

The Nagata dimension is not the key to understand the family of metric spaces whose Lipchitz free space is isomorphic to ℓ_1.

Conclusions

The Nagata dimension is not equivalent to the Separation modulus.

The Nagata dimension is not the key to understand the family of metric spaces whose Lipchitz free space is isomorphic to ℓ_1.

Is it the Separation modulus?

Conclusions

The Nagata dimension is not equivalent to the Separation modulus.

The Nagata dimension is not the key to understand the family of metric spaces whose Lipchitz free space is isomorphic to ℓ_1.

Is it the Separation modulus? More precisely: If $\mathcal{F}(M)$ is isomorphic to ℓ_1, is $\mathrm{SEP}(M)$ necessarily finite?

Thank you for your attention!