When Positivity Implies Compactness

Vinícius Miranda[1]

University of São Paulo

Summer School 2026: Topics in Banach Space Theory

[1]Supported by FAPESP (Grants 2023/12916-1 and 2025/08630-0) and Fapemig (Grant APQ-01853-23)

This talk is based on a joint work with Geraldo Botelho (UFU).

G. Botelho and V. C. C. Miranda, *Compact positive multilinear operators on Banach lattices*, Positivity **30** (2026), no. 2, Paper No. 20, 20 pp.

Motivation

Pitt's Theorem Every bounded linear operator from ℓ_p to ℓ_q is compact if and only if $1 \leq q < p < \infty$.

Pitt's Theorem Every bounded linear operator from ℓ_p to ℓ_q is compact if and only if $1 \leq q < p < \infty$.

A natural question is whether Pitt's Theorem persists when ℓ_q is replaced by $L_q(\nu)$.

Motivation

Pitt's Theorem Every bounded linear operator from ℓ_p to ℓ_q is compact if and only if $1 \leq q < p < \infty$.

A natural question is whether Pitt's Theorem persists when ℓ_q is replaced by $L_q(\nu)$.

The bounded linear operator $T : \ell_2 \to L_1([0,1])$ given by

$$T((a_j)_j) = \sum_{j=1}^{\infty} a_j r_j$$

is not compact.

Motivation

Pitt's Theorem Every bounded linear operator from ℓ_p to ℓ_q is compact if and only if $1 \leq q < p < \infty$.

A natural question is whether Pitt's Theorem persists when ℓ_q is replaced by $L_q(\nu)$.

The bounded linear operator $T : \ell_2 \to L_1([0,1])$ given by

$$T((a_j)_j) = \sum_{j=1}^{\infty} a_j r_j$$

is not compact.

However, positivity implies compactness...

(Chen, Wickstead – 1998) All positive operators $\ell_p \to L_q(\nu)$ and $L_p(\mu) \to \ell_q$ are compact whenever $q < p$.

Multilinear Pitt-type theorem

(Pełczyński - 1957) All continuous n-homogeneous polynomials $P : \ell_p \to \ell_q$ are compact if $nq < p$.

Multilinear Pitt-type theorem

(Pełczyński - 1957) All continuous n-homogeneous polynomials $P : \ell_p \to \ell_q$ are compact if $nq < p$.

(Alencar, Floret - 1997) Every continuous n-linear operator from $\ell_{p_1} \times \cdots \times \ell_{p_n}$ to ℓ_q is compact whenever $\sum_{i=1}^{n} \frac{1}{p_i} < \frac{1}{q}$.

Other related results were presented by V. Dimant and I. Zalduendo (1996); R. Gonzalo and J. A. Jaramillo (1993 and 1995)

We have two different extensions of Pitt's theorem:

- multilinear operators between sequence spaces;

Motivation

We have two different extensions of Pitt's theorem:

- multilinear operators between sequence spaces;
- positive operators involving L_p-spaces.

We have two different extensions of Pitt's theorem:

- multilinear operators between sequence spaces;
- positive operators involving L_p-spaces.

Question: What can be said about positive n-linear operators

$$\ell_{p_1} \times \cdots \times \ell_{p_n} \longrightarrow L_q(\nu)$$

and

$$L_{p_1}(\mu_1) \times \cdots \times L_{p_n}(\mu_n) \longrightarrow \ell_q?$$

We have two different extensions of Pitt's theorem:

- multilinear operators between sequence spaces;
- positive operators involving L_p-spaces.

Question: What can be said about positive n-linear operators

$$\ell_{p_1} \times \cdots \times \ell_{p_n} \longrightarrow L_q(\nu)$$

and

$$L_{p_1}(\mu_1) \times \cdots \times L_{p_n}(\mu_n) \longrightarrow \ell_q?$$

Strategy: Adapt Chen-Wickstead's technique to the multilinear setting.

Banach lattices

A **Banach lattice** is a real Banach space $(E, \|\cdot\|)$ endowed with a partial order $\leq$ such that:

- If $x \leq y$ in E, then $x + z \leq y + z$ and $\lambda x \leq \lambda y$ for every $z \in E$ and $\lambda \in \mathbb{R}^+$.
- For every $x, y \in E$, the supremum $x \vee y = \sup\{x, y\}$ and the infimum $x \wedge y = \inf\{x, y\}$ exist. In particular, for each $x \in E$,

$$|x| = x \vee (-x),\ x^+ = x \vee 0 \text{ e } x^- = (-x) \vee 0.$$

- If $|x| \leq |y|$, then $\|x\| \leq \|y\|$.

Banach lattices

A **Banach lattice** is a real Banach space $(E, \|\cdot\|)$ endowed with a partial order $\leq$ such that:

- If $x \leq y$ in E, then $x + z \leq y + z$ and $\lambda x \leq \lambda y$ for every $z \in E$ and $\lambda \in \mathbb{R}^+$.
- For every $x, y \in E$, the supremum $x \vee y = \sup\{x, y\}$ and the infimum $x \wedge y = \inf\{x, y\}$ exist. In particular, for each $x \in E$,

$$|x| = x \vee (-x),\ x^+ = x \vee 0 \text{ e } x^- = (-x) \vee 0.$$

- If $|x| \leq |y|$, then $\|x\| \leq \|y\|$.

An n-linear operator $T : E_1 \times \cdots \times E_n \to F$ is called **positive** if

$$T(x_1, \ldots, x_n) \geq 0 \ \text{ for all } \ x_1 \geq 0, \ldots, x_n \geq 0.$$

M-weakly compact operators

A bounded linear operator $T : E \to X$ is called **M-weakly compact** if $\|Tx_n\| \to 0$ for every disjoint sequence $(x_n)_n \subset B_E$.

M-weakly compact operators

A bounded linear operator $T : E \to X$ is called **M-weakly compact** if $\|Tx_n\| \to 0$ for every disjoint sequence $(x_n)_n \subset B_E$.

Let $A : E_1 \times \cdots \times E_n \to X$ be an n-linear operator.

M-weakly compact operators

A bounded linear operator $T : E \to X$ is called **M-weakly compact** if $\|Tx_n\| \to 0$ for every disjoint sequence $(x_n)_n \subset B_E$.

Let $A : E_1 \times \cdots \times E_n \to X$ be an n-linear operator.

• We say that A is M-**weakly compact** if

$$A(x_1^k, \ldots, x_n^k) \longrightarrow 0$$

in norm whenever each $(x_i^k)_k$ is a bounded disjoint sequence in E_i.

M-weakly compact operators

A bounded linear operator $T : E \to X$ is called **M-weakly compact** if $\|Tx_n\| \to 0$ for every disjoint sequence $(x_n)_n \subset B_E$.

Let $A : E_1 \times \cdots \times E_n \to X$ be an n-linear operator.
• We say that A is M-**weakly compact** if

$$A(x_1^k, \ldots, x_n^k) \longrightarrow 0$$

in norm whenever each $(x_i^k)_k$ is a bounded disjoint sequence in E_i.
• We say that A is **separately** M-**weakly compact** if, after fixing all variables but one, the resulting linear operator is M-weakly compact.

M-weakly compact operators

A bounded linear operator $T : E \to X$ is called **M-weakly compact** if $\|Tx_n\| \to 0$ for every disjoint sequence $(x_n)_n \subset B_E$.

Let $A : E_1 \times \cdots \times E_n \to X$ be an n-linear operator.
• We say that A is M-**weakly compact** if

$$A(x_1^k, \ldots, x_n^k) \longrightarrow 0$$

in norm whenever each $(x_i^k)_k$ is a bounded disjoint sequence in E_i.
• We say that A is **separately** M-**weakly compact** if, after fixing all variables but one, the resulting linear operator is M-weakly compact.
• A is said to be **strongly** M-**weakly compact** if, fixing any $k \in \{0, \ldots, n-1\}$ variables, the resulting $(n-k)$-linear operator is M-weakly compact.

Strongly M-weakly compact operators

The word **strongly** means that the M-weak compactness condition remains true after fixing some variables.

Strongly M-weakly compact operators

The word **strongly** means that the M-weak compactness condition remains true after fixing some variables.

For a bilinear operator $A : E_1 \times E_2 \longrightarrow F$, being strongly M-weakly compact is equivalent to:

A is M-weakly compact $\quad + \quad$ A is separately M-weakly compact.

Strongly M-weakly compact operators

The word **strongly** means that the M-weak compactness condition remains true after fixing some variables.

For a bilinear operator $A : E_1 \times E_2 \longrightarrow F$, being strongly M-weakly compact is equivalent to:

A is M-weakly compact $\quad + \quad$ A is separately M-weakly compact.

For a trilinear operator $A : E_1 \times E_2 \times E_3 \longrightarrow F$, we need more. For instance, for each fixed $x_1 \in E_1$, the bilinear map

$$(x_2, x_3) \longmapsto A(x_1, x_2, x_3)$$

must also be M-weakly compact.

Although the definition of strongly M-weakly compact operators may look restrictive, the following result shows that it holds automatically in many useful situations.

Strongly M-weakly compact operators

Although the definition of strongly M-weakly compact operators may look restrictive, the following result shows that it holds automatically in many useful situations.

Theorem Let $E_1, \ldots, E_n$ and F be Banach lattices such that

$$\sum_{i=1}^{n} \frac{1}{s(E_i)} < \frac{1}{\sigma(F)}.$$

(1) If F is atomic, then every $A \in \mathcal{L}(E_1, \ldots, E_n; F)$ is strongly M-weakly compact.

(2) Every $A \in \mathcal{L}^r(E_1, \ldots, E_n; F)$ is strongly M-weakly compact.

Lower and upper indices

The **lower index** $s(E)$ measures how well disjoint sums in E are controlled from above by ℓ_p-norms:

$$s(E) = \sup\left\{p \geq 1 : \left\|\sum_{i=1}^{n} x_i\right\| \leq M_p \|(x_1, \dots, x_n)\|_p\right\}.$$

The **upper index** $\sigma(E)$ measures how well disjoint sums in E are control ℓ_p-norms from below:

$$\sigma(E) = \inf\left\{p \geq 1 : \|(x_1, \dots, x_n)\|_p \leq M_p \left\|\sum_{i=1}^{n} x_i\right\|\right\}.$$

In both cases, the inequalities are required for all finite disjoint subsets $\{x_1, \dots, x_n\}$ of E and for some constant $M_p > 0$.

(1) Since $s(L_p(\mu)) = \sigma(L_p(\mu)) = p$ holds for every $1 \leq p < \infty$, we have that every continuous n-linear operator

$$A : L_{p_1}(\mu_1) \times \cdots \times L_{p_n}(\mu_n) \longrightarrow \ell_q$$

is strongly M-weakly compact, whenever $\sum_{i=1}^{n} \frac{1}{p_i} < \frac{1}{q}$. On the other hand, every regular n-linear operator

$$A : L_{p_1}(\mu_1) \times \cdots \times L_{p_n}(\mu_n) \longrightarrow L_q(\nu)$$

is strongly M-weakly compact, $\sum_{i=1}^{n} \frac{1}{p_i} < \frac{1}{q}$.

(2) For $1 < p < 2$, it is known that $s(\mathrm{FBL}[L_p(\mu)]) = p$. Therefore, if $1 < p_1, \ldots, p_n < 2$ satisfy $\sum_{i=1}^{n} \frac{1}{p_i} < \frac{1}{q}$, then every regular n-linear operator

$$A : \mathrm{FBL}[L_{p_1}(\mu_1)] \times \cdots \times \mathrm{FBL}[L_{p_n}(\mu_n)] \longrightarrow L_q(\nu)$$

is strongly M-weakly compact.

(2) For $1 < p < 2$, it is known that $s(\mathrm{FBL}[L_p(\mu)]) = p$. Therefore, if $1 < p_1, \dots, p_n < 2$ satisfy $\sum_{i=1}^n \frac{1}{p_i} < \frac{1}{q}$, then every regular n-linear operator

$$A : \mathrm{FBL}[L_{p_1}(\mu_1)] \times \cdots \times \mathrm{FBL}[L_{p_n}(\mu_n)] \longrightarrow L_q(\nu)$$

is strongly M-weakly compact.

The same holds for every continuous n-linear operator with range ℓ_q.

Main Results

Theorem Let $A : E_1 \times \cdots \times E_n \to F$ be a strongly M-weakly compact n-linear positive operator. If one of the following conditions holds, then A is compact:
(1) $E_1, \ldots, E_m$ are atomic with order continuous norms.
(2) F is atomic with order continuous norm.

Theorem Let $A : E_1 \times \cdots \times E_n \to F$ be a strongly M-weakly compact n-linear positive operator. If one of the following conditions holds, then A is compact:
(1) $E_1, \ldots, E_m$ are atomic with order continuous norms.
(2) F is atomic with order continuous norm.

Proof: Let $A : E_1 \times \cdots \times E_n \to F$ be a strongly M-weakly compact positive operator and let $\varepsilon > 0$ be given.

Main Results

Theorem Let $A : E_1 \times \cdots \times E_n \to F$ be a strongly M-weakly compact n-linear positive operator. If one of the following conditions holds, then A is compact:
(1) $E_1, \ldots, E_m$ are atomic with order continuous norms.
(2) F is atomic with order continuous norm.

Proof: Let $A : E_1 \times \cdots \times E_n \to F$ be a strongly M-weakly compact positive operator and let $\varepsilon > 0$ be given.

Lemma Let $A : E_1 \times \cdots \times E_n \to F$ be a strongly M-weakly compact positive n-linear operator. Then, for each $\varepsilon > 0$ there are $y_1 \in E_1^+, \ldots, y_n \in E_n^+$ such that

$$A(B_{E_1} \times \cdots \times B_{E_n}) \subset A([-y_1, y_1] \times \cdots \times [-y_n, y_n]) + \varepsilon B_F.$$

By the above lemma, there are $y_1 \in E_1^+, \dots, y_n \in E_n^+$ such that

$$A(B_{E_1} \times \cdots \times B_{E_n}) \subset A([-y_1, y_1] \times \cdots \times [-y_n, y_n]) + \varepsilon\, B_F.$$

By the above lemma, there are $y_1 \in E_1^+, \ldots, y_n \in E_n^+$ such that

$$A(B_{E_1} \times \cdots \times B_{E_n}) \subset A([-y_1, y_1] \times \cdots \times [-y_n, y_n]) + \varepsilon\, B_F.$$

(1) Suppose that $E_1, \ldots, E_m$ are atomic with order continuous norms. In this case, the order intervals $[-y_1, y_1], \ldots, [-y_n, y_n]$ are relatively compact in $E_1, \ldots, E_n$, respectively.

By the above lemma, there are $y_1 \in E_1^+, \dots, y_n \in E_n^+$ such that

$$A(B_{E_1} \times \cdots \times B_{E_n}) \subset A([-y_1, y_1] \times \cdots \times [-y_n, y_n]) + \varepsilon\, B_F.$$

(1) Suppose that $E_1, \dots, E_m$ are atomic with order continuous norms. In this case, the order intervals $[-y_1, y_1], \dots, [-y_n, y_n]$ are relatively compact in $E_1, \dots, E_n$, respectively. So, $A([-y_1, y_1] \times \cdots \times [-y_n, y_n])$ is relatively compact in F.

By the above lemma, there are $y_1 \in E_1^+, \ldots, y_n \in E_n^+$ such that

$$A(B_{E_1} \times \cdots \times B_{E_n}) \subset A([-y_1, y_1] \times \cdots \times [-y_n, y_n]) + \varepsilon\, B_F.$$

(1) Suppose that $E_1, \ldots, E_m$ are atomic with order continuous norms. In this case, the order intervals $[-y_1, y_1], \ldots, [-y_n, y_n]$ are relatively compact in $E_1, \ldots, E_n$, respectively. So, $A([-y_1, y_1] \times \cdots \times [-y_n, y_n])$ is relatively compact in F.

(2) Assume now that F is atomic with order continuous norm. Since A is positive, A is order bounded, so there exists $z \in F$ such that $A([-y_1, y_1] \times \cdots \times [-y_n, y_n]) \subset [-z, z]$.

By the above lemma, there are $y_1 \in E_1^+, \dots, y_n \in E_n^+$ such that

$$A(B_{E_1} \times \cdots \times B_{E_n}) \subset A([-y_1, y_1] \times \cdots \times [-y_n, y_n]) + \varepsilon B_F.$$

(1) Suppose that $E_1, \dots, E_m$ are atomic with order continuous norms. In this case, the order intervals $[-y_1, y_1], \dots, [-y_n, y_n]$ are relatively compact in $E_1, \dots, E_n$, respectively. So, $A([-y_1, y_1] \times \cdots \times [-y_n, y_n])$ is relatively compact in F.

(2) Assume now that F is atomic with order continuous norm. Since A is positive, A is order bounded, so there exists $z \in F$ such that $A([-y_1, y_1] \times \cdots \times [-y_n, y_n]) \subset [-z, z]$. Since $[-z, z]$ is relatively compact, we conclude that $A(B_{E_1} \times \cdots \times B_{E_n})$ is relatively compact in F. □

Main results

Corollary Let $1 < p_1, \ldots, p_n < \infty, 1 \leq q < \infty$ be such that $\sum_{i=1}^{n} \frac{1}{p_i} < \frac{1}{q}$. All positive n-linear operators

$$L_{p_1}(\mu_1) \times \cdots \times L_{p_n}(\mu_n) \to \ell_q$$

and

$$\ell_{p_1} \times \cdots \times \ell_{p_n} \to L_q(\nu)$$

are compact.

Corollary Let $1 < p_1, \ldots, p_n < \infty, 1 \leq q < \infty$ be such that $\sum\limits_{i=1}^{n} \frac{1}{p_i} < \frac{1}{q}$. All positive n-linear operators

$$L_{p_1}(\mu_1) \times \cdots \times L_{p_n}(\mu_n) \to \ell_q$$

and

$$\ell_{p_1} \times \cdots \times \ell_{p_n} \to L_q(\nu)$$

are compact.

Remark: (1) The bilinear map

$$A((a_j)_j, (b_j)_j) = \sum_{j=1}^{\infty} a_j b_j r_j,$$

defines a non-compact bilinear operator from $\ell_4 \times \ell_4$ to $L_1([0,1])$.

Corollary Let $1 < p_1, \ldots, p_n < \infty, 1 \leq q < \infty$ be such that $\sum_{i=1}^{n} \frac{1}{p_i} < \frac{1}{q}$. All positive n-linear operators

$$L_{p_1}(\mu_1) \times \cdots \times L_{p_n}(\mu_n) \to \ell_q$$

and

$$\ell_{p_1} \times \cdots \times \ell_{p_n} \to L_q(\nu)$$

are compact.

Remark: (2) By Chen and Wickstead's work, there exists a positive non-compact operator $T : L_2([0,1]) \to L_1([0,1])$. Then,

$$B(f, a) = \varphi(a)T(f)$$

defines a non-compact positive bilinear operator from $L_2([0,1]) \times \ell_3$ to $L_1([0,1])$, where $\varphi : \ell_3 \to \mathbb{R}$ is a positive linear functional functional.

Corollary Let $n \in \mathbb{N}$ and $1 \leq p, q < \infty$ be such that $nq < p$. Then, every positive n-homogeneous polynomial $P : L_p(\mu) \to \ell_q$ is compact. In particular, $\mathcal{P}^r({}^n L_p(\mu); \ell_q)$ is a KB-space.

G. Botelho, V. C. C. Miranda and P. Rueda, *Banach lattices of homogeneous polynomials not containing* c_0, arXiv:2312.11717v3

Applications

Corollary Let $1 < p_1, \ldots, p_n < \infty, 1 \leq q < \infty$ be such that $\sum_{i=1}^{n} \frac{1}{p_i} < \frac{1}{q}$. Then, all positive n-linear operators

$$\ell_{p_1} \times \cdots \times \ell_{p_n} \to L_q(\nu)$$

are norm attaining.

L. A. Garcia, J. L. P. Luiz and V. Miranda, *Norm attainment for multilinear operators and polynomials on Banach Spaces and Banach lattices*, arXiv:2605.12117

Corollary Let $1 < p_1, \ldots, p_n < \infty, 1 \leq q < \infty$ be such that $\sum_{i=1}^{n} \frac{1}{p_i} < \frac{1}{q}$. Then, all positive n-linear operators

$$\ell_{p_1} \times \cdots \times \ell_{p_n} \to L_q(\nu)$$

are norm attaining.

L. A. Garcia, J. L. P. Luiz and V. Miranda, *Norm attainment for multilinear operators and polynomials on Banach Spaces and Banach lattices*, arXiv:2605.12117

Corollary Let $n \in \mathbb{N}$ and $1 \leq p, q < \infty$ be such that $nq < p$. Then, every positive n-homogeneous polynomial $P : \ell_p \to L_q(\mu)$ is norm-attaining.

Thank you all for your attention!