Semiprojective Banach lattices

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Absolute neighborhood retract

A metric space Y is an absolute neighborhood retract (ANR) if for any embedding $Y \subset Z$ as a closed subspace in a metric space Z, there exist a neighborhood X of Y in Z and Y is a retract of X.

ARs and ANRs

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Order ideal of a Banach lattice

Given a Banach lattice E we say that a linear subspace $J \subset E$ is an order (lattice) ideal of E if for any $x, y \in E$ we have $y \in J$ implies $|y| \in J$ and $\mathbf{0}_E \preceq x \preceq y$ and $y \in J$ implies $x \in J$.

Ideals and order units

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Principal order ideal

We say that J_u is a principal order ideal generated by a vector $u \in E$, where $u \succeq \mathbf{0}_E$ and $u \neq \mathbf{0}_E$, if

$$J_u = \{y \in E : \exists r > 0 : |y| \preceq ru\}.$$

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Topological order unit

An element e of Banach lattice E is called a topological order unit if J_e is dense in E.

Projectivity of C^*-algebras

Projective C^*-algebra

C^*-algebra $\mathcal{A}$ is called projective if for any C^*-algebra $\mathcal{A}_0$ and closed ideal $\mathcal{I} \subset \mathcal{A}$ and any $*-$homomorphism $f\colon \mathcal{A} \to \mathcal{A}_0/\mathcal{I}$ there exist a lift $\widetilde{f}\colon \mathcal{A} \to \mathcal{A}_0$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathcal{A}_0 \arrow[d, "Q"] \\ \mathcal{A} \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & \mathcal{A}_0/\mathcal{I} \end{tikzcd}$$

where Q is quotient map.

Inductive limit ideal

An ideal $\mathcal{I}$ in a C^*-algebra $\mathcal{A}$ is an inductive limit ideal if there is an increasing sequence $\mathcal{I}_n$ of closed subideals of $\mathcal{I}$ with dense union that is $\mathcal{I} = \overline{\bigcup_{n=1}^{\infty} \mathcal{I}_n}$.

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Semiprojective C^*-algebra

A separable C^*-algebra $\mathcal{A}$ is called semiprojective if, for any separable C^*-algebra $\mathcal{A}_0$ and closed inductive limit ideal $\mathcal{I} \subset \mathcal{A}$ and any $*-$homomorphism $f\colon \mathcal{A} \to \mathcal{A}_0/\mathcal{I}$ there exist $n \in \mathbb{N}$ and a map $\widetilde{f}\colon \mathcal{A} \to \mathcal{A}_0/\mathcal{I}_n$ such that the following diagram commutes:

$$\begin{tikzcd}
& \mathcal{A}_0/\mathcal{I}_n \arrow[d, "\pi_n"] \\
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\end{tikzcd}$$

Projectivity and semiprojectivity of commutative C^*-algebras

Theorem 1 (Chigogidze, Dranishnikov, 2010)

Let X be a compact, metric space. Then $C(X;\mathbb{C})$ is projective C^*-algebra if and only if X is an absolute retract with $\dim(X) \leqslant 1$.

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Theorem 2 (Sørensen, Thiel, 2012)

Let X be a compact, metric space. Then $C(X;\mathbb{C})$ is semiprojective C^*−algebra if and only if X is an absolute neighborhood retract with $\dim(X) \leqslant 1$.

Projectivity of Banach lattices

1^+-projective Banach lattice

Banach lattice X is called projective if, given any Banach lattice Y, a closed order ideal $J \subset Y$, for every lattice homomorphism $S\colon X \to Y/J$ and all $\varepsilon \in (0, \infty)$, there exists a lattice homomorphism $\widetilde{S}\colon X \to Y$ such that the following diagram commutes

$$\begin{tikzcd} & Y \arrow[d, "Q"] \\ X \arrow[ur, "\widetilde{S}"] \arrow[r, "S"'] & Y/J \end{tikzcd}$$

where Q is again quotient map. Moreover $\|\widetilde{S}\|_{\mathcal{B}(X,Y)} \leqslant (1+\varepsilon)\|S\|_{\mathcal{B}(X,Y/J)}$.

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Thoerem 3 (Avilés, Martinez-Cervantes, Rodriguez-Abellan, 2020)

Let X be a compact, metric space. Then $C(X;\mathbb{R})$ is 1^+−projective Banach lattice if and only if X is an absolute neighborhood retract.

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Semiprojective Banach lattice

Banach lattice X is called semiprojective if for any separable Banach lattice Y and closed inductive limit ideal $J \subset Y$ and any lattice homomorphism $f\colon X \to Y/J$ there exist $n \in \mathbb{N}$ and, for every $\varepsilon \in (0, \infty)$, a map $\widetilde{f}\colon X \to Y/J_n$ such that the following diagram commutes:

$$\begin{tikzcd} & Y/J_n \arrow[d, "\pi_n"] \\ X \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & Y/J \end{tikzcd}$$

Moreover $\|\widetilde{S}\|_{\mathcal{B}(X,Y/J_n)} \leqslant (1+\epsilon)\|S\|_{\mathcal{B}(X,Y/J)}$.

Semiprojectivity of $C(K)$ lattices

Theorem 4 (Kania, N.)

Let K be a compact metric space. If K is not an ANR, then $C(K;\mathbb{R})$ is not semiprojective.

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Corollary 1

For a compact metric space K a Banach lattice $C(K;\mathbb{R})$ is semiprojective iff it is projective.

Theorem 5 (Kania, N.)

Let Γ be a set and let $(P_\gamma)_{\gamma\in\Gamma}$ be a family of 1^+-projective Banach lattices, each admitting a topological order unit. Then the Banach lattice

$$P := \Big(\bigoplus_{\gamma\in\Gamma} P_\gamma\Big)_{\ell_1(\Gamma)}$$

is semiprojective. If Γ is countable, then P is 1^+-projective.

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Corollary 2

For every set Γ, the Banach lattice $\ell_1(\Gamma)$ is semiprojective. It is 1^+-projective if and only if Γ is countable.

Theorem 6 (Kania, N.)

Let A be a semiprojective Banach lattice, let X be a separable Banach lattice, let $J \subseteq X$ be a closed ideal, and let $T\colon A \to X/J$ be a contractive lattice homomorphism. Then there exists a lattice homomorphism $\widehat{T}\colon A \to X$ such that $Q \circ \widehat{T} = T$, where $Q\colon X \to X/J$ is the quotient map.

Further properties of semiprojectivity

Theorem 6 (Kania, N.)

Let A be a semiprojective Banach lattice, let X be a separable Banach lattice, let $J \subseteq X$ be a closed ideal, and let $T\colon A \to X/J$ be a contractive lattice homomorphism. Then there exists a lattice homomorphism $\widehat{T}\colon A \to X$ such that $Q \circ \widehat{T} = T$, where $Q\colon X \to X/J$ is the quotient map.

Theorem 7 (Kania, N.)

Let E be a non-zero separable Banach lattice such that $\mathrm{Hom}(E, \mathbb{R}) = \{0\}$. Then E is not semiprojective.

Classical nonexamples

Corollary 3

Let Γ be an infinite set. Then $c_0(\Gamma)$ is not semiprojective. If $1 < p < \infty$, then $\ell_p(\Gamma)$ is not semiprojective either.

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For every $1 \leqslant p < \infty$, the Banach lattice $L_p([0,1])$ is not semiprojective.

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Corollary 4

For every $1 \leqslant p < \infty$, the Banach lattice $L_p([0,1])$ is not semiprojective.

Corollary 5

Let (Ω, Σ, μ) be an atomless measure space and let Φ be an Orlicz function. Denote by $H^\Phi(\mu)$ the corresponding Orlicz heart, equipped with the Luxemburg norm. If $H^\Phi(\mu)$ is non-zero and separable, then $H^\Phi(\mu)$ is not semiprojective. If moreover $L^\Phi(\mu)$ is non-zero and separable, then $L^\Phi(\mu)$ is not semiprojective.

Open questions

Question 1

Let

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

be a short exact sequence of Banach lattices. If J and B are semiprojective, must A be semiprojective?

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Let X be a separable Banach lattice. Does semiprojectivity of X imply that X is 1^+-projective? Equivalently, do semiprojectivity and 1^+-projectivity coincide for separable Banach lattices?

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Question 3

Is every separable Banach lattice an inductive limit of semiprojective Banach lattices?

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Thank you for attention.