Countable tensor products and similarity to contraction semigroups

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Overview

1. Similarity to contraction
2. Operator semigroups
3. (Infinite) Tensor products

Operators similar to contractions

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Sz.-Nagy's dilation theorem '53: If $\|A\| \leq 1$, then $\exists$ a Hilbert space K with $H \hookrightarrow K$ isometrically, a unitary $U \in \mathcal{L}(K)$ s.t.

$$U = \begin{pmatrix} * & * \\ 0 & A \end{pmatrix}, \qquad K = H^{\perp} \oplus H \implies A^n = PU^n|_H, \qquad n \in \mathbb{N},$$

where $P : K \to H$ is the orthogonal projection.

If $\|A\| \leq 1$, then von Neumann's inequality ('51) gives

$$\|p(A)\| \leq \sup_{z \in \mathbb{D}} |p(z)|, \qquad p \in \mathbb{C}[z].$$

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However, the converses fail in general:

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- **Pisier '96**: polynomially bounded need not imply similar to a contraction (Halmos '70 "*Ten problems in Hilbert space*").

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The correct general characterization is complete polynomial boundedness (Paulsen '82)

$$A \text{ similar to a contraction} \iff A \text{ completely polynomially bounded.}$$

A C_0-semigroup on a Hilbert space H is a family

$$(T(t))_{t\geq 0} \subset \mathcal{L}(H)$$

such that

- $T(0) = I$ and $T(t+s) = T(t)T(s)$;
- the mapping $t \mapsto T(t)$ is continuous in the strong operator topology.

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Applications: heat, transport, Schrödinger and wave equations; control theory, probability and dynamical systems.

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Applications: **transport equation**

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$$u(t) = T(t)h = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(\cdot - y)^2}{4t}} h(y)\, dy, \quad t \geq 0.$$

Applications: **heat equation**

Similarity to contraction semigroups

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$$\|Q^{-1}T(t)Q\| \leq 1, \qquad t \geq 0.$$

Equivalently, $(T(t))_{t\geq 0}$ is contractive for some **equivalent** **Hilbertian** norm $\|\cdot\|_{\mathrm{eq}}$ on H

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As before, contractive C_0-semigroups admit unitary dilations $(U(t))_{t\in\mathbb{R}}$

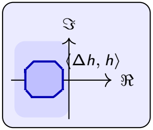
$$H \hookrightarrow K, \qquad T(t) = PU(t)|_H, \qquad t \geq 0.$$

Several criteria based on the generator

Let $\Delta := \lim\limits_{t\to 0^+} \dfrac{T(t)-I}{t}$ be the generator of $(T(t))_{t\geq 0}$.

- **Lumer–Phillips '61:**

$$(T(t))_{t\geq 0} \text{ contractive} \quad \Longleftrightarrow \quad \Delta \text{ maximal dissipative}.$$



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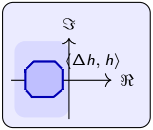

- **Foias–Sz.-Nagy '60:** if

$$V := (\Delta + I)(\Delta - I)^{-1},$$

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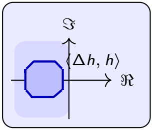

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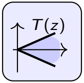

- **Functional calculus:** if $(T(t))_{t\geq 0}$ is analytic, then TFAE:
  - $(T(t))_{t\geq 0}$ is similar to a contraction semigroup.
  - Δ has bounded H^∞-calculus. (McIntosh '86)
  - Δ has bounded imaginary powers. (Le Merdy '98)

Operators versus semigroups

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Single operator: Let $A \in \mathcal{L}(H)$. Then

A power bounded and compact $\implies$ A is similar to a contraction (Sz.-Nagy '59)

$r(A) < 1$ $\implies$ A is similar to a strict contraction (Rota's theorem '60)

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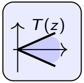

Semigroups:
Le Merdy '00 constructed an **analytic** C_0-semigroup $(T(t))_{t\geq 0}$ such that

$$\|T(t)\| \leq Me^{-\omega t},\ t \geq 0 \quad \text{(for some } M, \omega > 0\text{);}$$
$$T(t) \text{ is compact for all } t > 0;$$

$\iff r(T(t)) < 1$

but is **NOT** similar to a contraction semigroup (see also Packel '69, Chernoff '76, and Baillon & Clement '91).

Towards stronger counterexamples: (quasi-)nilpotency

A C_0-semigroup $(T(t))_{t\geq 0}$ is said to be:

1. quasi-nilpotent if for every $\omega > 0$, there exists $M_\omega > 0$ such that
$$\|T(t)\| \leq M_\omega e^{-\omega t}, \ t \geq 0 \quad \boxed{\iff \ r(T(t)) = 0}$$
2. nilpotent if there exists $\tau > 0$ such that $T(\tau) = 0$ ($\iff$ $T(t) = 0$ for all $t \geq \tau$).
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Connection with Δ^{-1}-problem: If $\sup_{t\geq 0} \|T(t)\| < \infty$ and Δ injective, does Δ^{-1} generate a bounded C_0-semigroup? **Problem 1** in Madou, Remizov & Vafadar '24 "*Open problems in one-parameter operator semigroups theory*"

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Theorem (O.-M. & Tomilov)

- *There exists a C_0-semigroup $(T(t))_{t\geq 0}$* **nilpotent** *and* **immediately compact** *which is* **NOT** *similar to a contraction semigroup.*
- *There exists an* **analytic** *C_0-semigroup $(T(t))_{t\geq 0}$ that is* **quasi-nilpotent** *and* **immediately compact** *which is* **NOT** *similar to a contraction semigroup.*

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Main tool for the construction of our examples: **tensor products** of Hilbert spaces

Finite tensor products

Let $H_1, \ldots, H_N$ be Hilbert spaces. We first consider the algebraic tensor product

$$H_1 \otimes_{\mathrm{alg}} \cdots \otimes_{\mathrm{alg}} H_N,$$

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Example: if Ω, Λ measurable spaces, and H a Hilbert space

$$L^2(\Omega) \bigotimes L^2(\Lambda) \simeq L^2(\Omega \times \Lambda), \qquad L^2(\Omega) \bigotimes H \simeq H\text{-valued } L^2(\Omega) \text{ functions.}$$

Tensor product of operators

If $A_1 \in \mathcal{L}(H_1), \dots, A_N \in \mathcal{L}(H_N)$, then

$$\left(\bigotimes_{n=1}^{N} A_n\right)(\otimes_{n=1}^{N} h_n) := \otimes_{n=1}^{N} A_n h_n, \qquad h_1 \in H_1, \dots h_n \in H_n$$

extends to a bounded operator on $\bigotimes_{n=1}^{N} H_n$ satisfying

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Theorem (Paulsen, Pearcy & Petrovic '95)

Let $A_1 \in \mathcal{L}(H_1)$, $A_2 \in \mathcal{L}(H_2)$. Then

$$A_1 \bigotimes A_2 \qquad \textit{is similar to a contraction}$$

if and **only if** *there exists $\alpha > 0$ such that*

$$\alpha A_1,\ \frac{1}{\alpha} A_2 \qquad \textit{are similar to contractions.}$$

Tensor products of semigroups

The equality $\|A \otimes B\| = \|A\|\|B\|$ implies that if $\lambda \in \mathbb{R}$ is such that

$$(e^{\lambda t} T(t))_{t\geq 0} \text{ and } (e^{-\lambda t} S(t))_{t\geq 0} \text{ are similar to contraction semigroups,} \tag{S}$$

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Theorem (O.-M., Tomilov)

Let $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ be two C_0-semigroups. Then

$$(T(t) \otimes S(t))_{t \geq 0} \qquad \textit{is similar to a contraction semigroup}$$

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Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be two C_0-semigroups. Then

$$(T(t) \otimes S(t))_{t\geq 0} \qquad \textit{is similar to a contraction semigroup}$$

*if and **only if** there exists $\lambda \in \mathbb{R}$ s.t. (S) holds.*

Then, the nilpotency of our counterexample is obtained by taking semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ such that

1. $(T(t))_{t\geq 0}$ is **nilpotent**;
2. there is **NO** $\lambda \in \mathbb{R}$ such that $(e^{-\lambda t} S(t))_{t\geq 0}$ is similar to a contraction semigroup. (Chernoff '76)

Infinite tensor products: incomplete tensor products

We recall von Neumann's construction ('39) of infinite tensor products of Hilbert spaces. Let $(H_n)_{n\in\mathbb{N}}$ be Hilbert spaces and let

$$\mathbf{e} = (e_n)_{n\in\mathbb{N}}, \qquad \|e_n\| = 1,$$

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$$\left\langle \bigotimes_{n\in\mathbb{N}} h_n, \bigotimes_{n\in\mathbb{N}} k_n \right\rangle = \prod_{n\in\mathbb{N}} \langle h_n, k_n \rangle.$$

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The product is actually finite, because almost all factors are 1.
The completion is the **incomplete tensor product**

$$\bigotimes_{n\in\mathbb{N}}^{\mathbf{e}} H_n.$$

Infinite tensor products: complete tensor products

The incomplete tensor product depends on the reference sequence $\mathbf{e} = (e_n)_{n\in\mathbb{N}}$.

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Different choices of $\mathbf{e}$ may produce mutually orthogonal "sectors". To collect all of them, one introduces an equivalence relation on reference sequences:

$$\mathbf{e} \sim \mathbf{f} \quad \Longleftrightarrow \quad \sum_{n\in\mathbb{N}} |1 - \langle e_n, f_n\rangle| < \infty \quad \Longleftrightarrow \quad \prod_{n\in\mathbb{N}} \langle e_n, f_n\rangle \text{ converges unconditionally.}$$

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The **complete tensor product** is the Hilbert direct sum of all incomplete tensor products:

$$\bigotimes_{n\in\mathbb{N}} H_n = \bigoplus_{[\mathbf{e}]} \overset{\mathbf{e}}{\bigotimes_{n\in\mathbb{N}}} H_n.$$

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- incomplete tensor product = one sector fixed by a reference vector;
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Thus:

- incomplete tensor product = one sector fixed by a reference vector; $\bigotimes_{n\in\mathbb{N}}^{\mathbf{e}} \mathbb{C} \simeq \mathbb{C}$.
- complete tensor product = the orthogonal sum of all sectors; $\bigotimes_{n\in\mathbb{N}} \mathbb{C}$ is **not** separable.

The tensor product norm is a **cross-norm** in both cases.

Why infinite tensor products?

- Classical motivation: mathematical physics, infinite quantum systems, operator algebras.
- Example: if (X, μ) is a probability space, then

$$\bigotimes_{n\in\mathbb{N}}^{\mathbf{1}} L^2(X,\mu) \simeq L^2(X^{\mathbb{N}}, \mu^{\mathbb{N}}),$$

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- Infinite tensoring is more delicate than finite tensoring:

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- Existing theory for non-unitary operators and semigroups is comparatively limited.

This motivates a framework adapted to general semigroups, where norm growth and C_0-regularity become part of the problem.

Similarity on infinite tensor products

Let $(T(t))_{t\geq 0}$ be similar to a contraction semigroup. Its similarity constant is defined as

$$\mathcal{C}\,(T(t))_{t\geq 0} := \inf\left\{\|Q\|\|Q^{-1}\| \,:\, Q\in\mathcal{L}(H) \text{ s.t. } \|Q^{-1}T(t)Q\|\leq 1,\ t\geq 0\right\}$$

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Theorem (O.-M., Tomilov)

For $n\in\mathbb{N}$, let $(T_n(t))_{t\geq 0}$ be a semigroup on H_n (no measurability assumed) such that

$$\left(\bigotimes_{n\in\mathbb{N}} T_n(t)\right)_{t\geq 0} \qquad \text{is well-defined and not identically zero,}$$

either in the complete sense or in the incomplete sense. Then

$$\left(\bigotimes_{n\in\mathbb{N}} T_n(t)\right)_{t\geq 0} \qquad \text{is similar to a contraction semigroup}$$

if and only if *there exists a sequence $(d_n)_{n\in\mathbb{N}}\in\ell^1(\mathbb{N})$ with $\sum_{n\in\mathbb{N}} d_n = 0$ such that*

$$(e^{d_n t}T_n(t))_{t\geq 0} \text{ is similar to a contraction} \qquad (n\in\mathbb{N})$$

and, if F runs over all finite subsets of $\mathbb{N}$, then

$$\sup_F \mathcal{C}\left(\bigotimes_{n\in F} e^{d_n t}T_n(t)\right)_{t\geq 0} < \infty.$$

Note on similarity constants

We have

$$\prod_{n\in\mathbb{N}} \mathcal{C}(e^{d_n t} T_n(t))_{t\geq 0} < \infty$$

$$\Downarrow$$

$$\sup_F \mathcal{C}\left(\bigotimes_{n\in F} e^{d_n t} T_n(t)\right)_{t\geq 0} < \infty$$

$$\Downarrow$$

$$\sup_{n\in\mathbb{N}} \mathcal{C}(e^{d_n t} T_n(t))_{t\geq 0} < \infty,$$

where F runs over all finite subsets of $\mathbb{N}$.

The above implications **CANNOT** be reversed, even for $H_n = \mathbb{C}^2$ and $(T_n(t))_{t\geq 0}$ with

$$\left(\bigotimes_{n\in\mathbb{N}} T_n(t)\right)_{t\geq 0} \quad \text{is well-defined and uniformly bounded.}$$

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Thank You!