Finite-dimensional Transportation Cost Spaces

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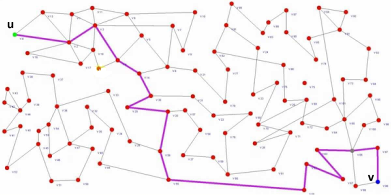
- Topic 2: Banach space of transportation problems with the norm induced by optimal transport: Names: Transportation cost, Kantorovich-Rubinstein norm, Optimal transport, Wasserstein norm.
- Topic 3: Free linear spaces generated by metric spaces with the norm induced by the metric: Arens-Eells spaces, free Banach spaces, Lipschitz-free spaces.

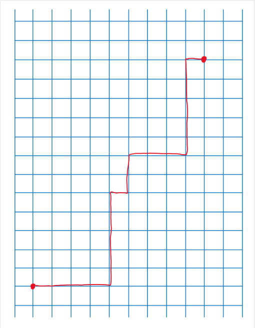
Finite metric spaces

We study finite metric spaces (X, d).



Without loss of generality, we may assume:

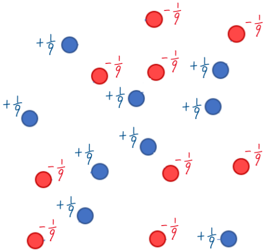
- X is a finite weighted graph
- $d(u, v)$ is shortest-path distance



Important example: planar grid $(X = [n]^2)$ with Manhattan metric

1. Transportation problem

A signed measure μ on (X, d) with total mass 0.



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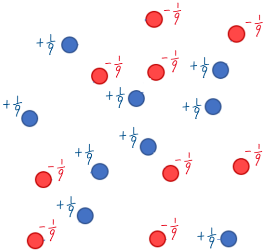
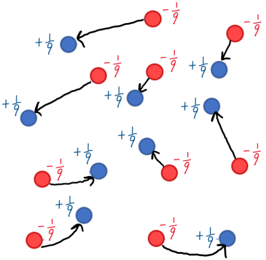
A representation of the form

$$\mu = \sum_{i=1}^{n} a_i(\mathbf{1}_{x_i} - \mathbf{1}_{y_i}), \quad a_i \geq 0.$$





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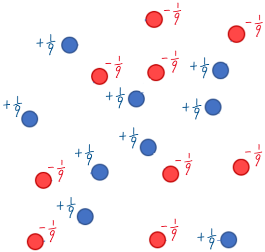
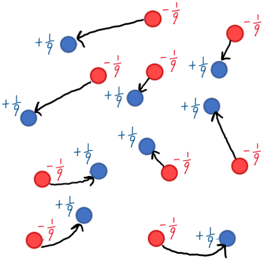
3. Transportation cost (TC) of the plan $\sum_{i=1}^{n} a_i\, d(x_i, y_i)$;
$\|\mu\|_{\mathrm{TC}} = \inf_{\text{plans}} \sum_{i=1}^{n} a_i\, d(x_i, y_i)$





Transportation cost (more formal presentation)

- Let (X, d) be a finite metric space. Consider a real-valued measure μ on X satisfying $\mu(X) = 0$. A natural and important interpretation of such a real measure is considering it as a *transportation problem*: one needs to transport certain product from points where $\mu(v) > 0$ to points where $\mu(v) < 0$.

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- One can easily see that a transportation problem μ can be represented as

$$\mu = a_1(\mathbf{1}_{x_1} - \mathbf{1}_{y_1}) + a_2(\mathbf{1}_{x_2} - \mathbf{1}_{y_2}) + \dots + a_n(\mathbf{1}_{x_n} - \mathbf{1}_{y_n}), \quad (1)$$

where $a_i \geq 0$, $x_i, y_i \in X$, and $\mathbf{1}_u(x)$ for $u \in X$ is the unit measure at u.

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where $a_i \geq 0$, $x_i, y_i \in X$, and $\mathbf{1}_u(x)$ for $u \in X$ is the unit measure at u.
- We call each such representation a *transportation plan* for μ, it can be interpreted as plan of moving a_i units of the product from x_i to y_i.

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- **Exercise:** Show that $\|\cdot\|_{\mathrm{TC}}$ is a norm.
- **Remark:** I advise you to do the exercises if you want to **start** doing research in this area.

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- **Disclaimer:** In this mini-course I do not even touch (at least) three important things: (a) Isometric classification developed by Alexander, Fradelizi, García-Lirola, and Zvavitch [AFGZ21], (b) Stochastic bases studied by Medina and Tresch [MT26+], and (c) Combinatorial theory initiated by Vershik [Ver15] (see [OO22] for more references).

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- **Exercise 3:** The spaces ℓ_1^2 and ℓ_∞^2 are isometric.

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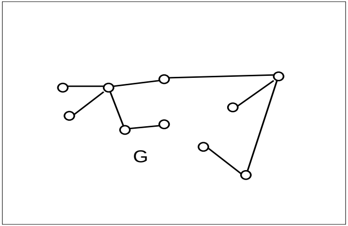

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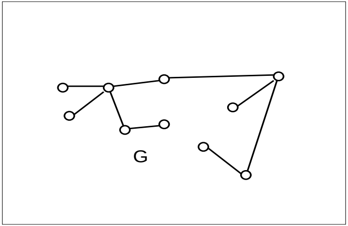

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- **Exercise 4:** Let T_n be a tree with n vertices. We allow tree to be *weighted*. This means that edges can be of any positive lengths. We endow the set of vertices of a tree with the metric defined as the length of the path between vertices. In a tree such path is unique.
  Prove that $\mathrm{TC}(T_n)$ is isometric to ℓ_1^{n-1}.
  (We shall need this fact.)

Additional Problems and Exercises

- **More difficult Exercise:** (Converse to the exercise on trees). If a finite metric space X is such that $\mathrm{TC}(X) = \ell_1^{n-1}$, then X is isometric to a (possibly weighted) tree with n vertices.

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- **(Relevant) Theorem:** Let G be a weighted graph for which $\mathrm{TC}(G)$ contains an isometric copy of ℓ_∞^k. Then G contains vertices whose degrees are at least 2^{k-2}. Furthermore, the number of such vertices should be nontrivially large, namely, if $\{e_i\}_{i=1}^k \subset \mathrm{TC}(X)$ are isometrically equivalent to the unit vector basis of ℓ_∞^k, then each vertex contained in a support of any of $\{e_i\}_{i=1}^k$ should have degree at least 2^{k-2}. [OO22]

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- **Theorem [AFGZ21, Corollary 3.8]:** The space $\mathrm{TC}(X)$ cannot be isometric to ℓ_∞^n for $n \geq 4$.

- **Exercise (with a hint):** The space $\mathrm{TC}(K_{4,4})$ contains ℓ_∞^4 isometrically [DKO21].

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- **Hint:** Let $\{a, b, c, d\}$ and $\{e, f, g, h\}$ be a bipartition. Use the basis:

	a	b	c	d	e	f	g	h
f_1	1/2	-1/2	1/2	-1/2	0	0	0	0
f_2	1/2	1/2	-1/2	-1/2	0	0	0	0
f_3	0	0	0	0	1/2	-1/2	1/2	-1/2
f_4	0	0	0	0	1/2	1/2	-1/2	-1/2

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$$\mathrm{TC}(G) = \ell_{1,d}(E(G))/Z(G),$$

where G is a connected weighted graph, $\ell_{1,d}(E(G))$ is a weighted ℓ_1 on $E(G)$, the subspace $Z(G)$ is a cycle space in $\ell_{1,d}(E(G))$, and Y/Z denotes the quotient space of a Banach space over its subspace. The first version of this result is [Ost13, Proposition 10.10]; further versions are in [OO20, AFGZ21, DKO21, OO22, SV26].

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  - *Weighted* means that each edge uv is assigned the *length function* $d_G(u, v) > 0$, and that the formula $d_G(u, v) =$ (the total weight of the shortest (no direction) path joining u and v) extends the length function to all pairs of vertices .

- A graph is called *unweighted* if the length function is equal to 1 on all edges.

- **Definition:** For finite-dimensional Banach spaces X and Y, we say that X is a *quotient space of* Y, if there is a linear operator $Q : Y \to X$ such that $Q(B(Y)) = B(X)$, where $B(X)$ denotes the *unit ball*, that is, $B(X) = \{x \in X : \|x\| \leq 1\}$. Let U be the kernel of Q, we write $X = Y/U$. The operator Q is called a *quotient map*.

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- **Important observations (A):** The quotient of a (finite-dimensional) polyhedral space is also polyhedral.
- **(B)** For each $x \in X$,

$$\|x\| = \inf\{\|y\| : Qy = x.\}$$

This is true because the equality $Q(B(Y)) = B(X)$ implies that the equality in the previous line holds if $\|x\| = 1$.

- We introduce the *weighted* ℓ_1*-space* $\ell_{1,d}(E(G))$ on the edge set of a weighted graph $E(G)$. It consists of real-valued functions $F : E(G) \to \mathbb{R}$ with the norm

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- The space $\ell_{1,d}(E(G))$ is polyhedral because its unit ball is the convex hull of functions $\pm f_e$, where f_e is equal to 0 on all edges except e, and equal to $\frac{1}{d_G(u,v)}$ if $e = uv$.

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- The space $\ell_{1,d}(E(G))$ is polyhedral because its unit ball is the convex hull of functions $\pm f_e$, where f_e is equal to 0 on all edges except e, and equal to $\frac{1}{d_G(u,v)}$ if $e = uv$.
- It is well-known that ℓ_p^n, $1 < p < \infty$, $n \geq 2$ and their subspaces of dimension ≥ 2, are not polyhedral.

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- It is well-known that ℓ_p^n, $1 < p < \infty$, $n \geq 2$ and their subspaces of dimension ≥ 2, are not polyhedral.
- Therefore the equality $\mathrm{TC}(G) = \ell_{1,d}(E(G))/Z(G)$ implies that $\mathrm{TC}(X)$ cannot contain isometric copies of ℓ_p^n, $1 < p < \infty$, $n \geq 2$ and their subspaces of dimension ≥ 2.

- The complete formal proof of $\mathrm{TC}(G) = \ell_{1,d}(E(G))/Z(G)$ looks rather complicated, but the main idea is simple: one can easily guess a suitable linear map $Q : \ell_{1,d}(E(G))) \to \mathrm{TC}(X)$, and then verify that it is actually a quotient map, and its kernel is the *cycle space* as it is defined in Algebraic Graph Theory, usually denoted $Z(G)$.

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- It looks like there is only one natural candidate for Q: We derive from any real-valued function F on $E(G)$ the following transportation plan.
- Let $\vec{e} = uv$. If $F(e) > 0$, we transport $F(e)$ units of the product from u to v. If $F(e) < 0$, we transport $-F(e) = |F(e)|$ unit of the product from v to u. The corresponding plan (as any transportation plan) can be regarded as a transportation problem. We let this transportation problem be $Q(F)$.

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- **Important:** One can easily verify that the cost of this transportation plan is $\|F\|_{1,d} := \sum_{uv \in E(G)} |F(uv)| d_G(u, v)$.

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- If $x_i y_i$ were edges, it would immediately lead to the corresponding function F. For non-adjacent pairs x_i, y_i, we replace $a_i(\mathbf{1}_{x_i} - \mathbf{1}_{y_i})$ with a more detailed plan along the shortest path joining x_i and y_i. Such plans (combined for all x_i, y_i) lead $F \in \ell_{1,d}(E(G))$ satisfying $QF = \mu$ and $\|F\|_{1,d} = \|\mu\|_{\mathrm{TC}}$.

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- So Q is the quotient map.

- **(b)** To determine the kernel of Q, we observe the following formula for Q. Let $F \in \ell_{1,d}(E(G))$, we have

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- **Definition:** Let $Z = \ker Q$. This subspace of $\ell_{1,d}(E(G))$ is called the *cycle space*. Actually, the cycle spaces is usually considered as the linear space of functions on $E(G)$, without any norm.
- The name 'cycle space' is chosen because this subspace is spanned by the set of signed indicator functions of all cycles (see the next slide).

- Now we consider a cycle C in G (in a graph-theoretical sense). We consider one of the two possible orientations of C satisfying the following condition: each vertex of C is a head of exactly one edge and a tail of exactly one edge. Now we introduce the *signed indicator function* $\chi_C \in \mathbb{R}^E$ of the oriented cycle C (in an oriented graph G) by

$$\chi_C(e) = \begin{cases} 1 & \text{if } e \in C \text{ and its orientations in } C \text{ and } G \text{ coincide} \\ -1 & \text{if } e \in C \text{ but its orientations in } C \text{ and } G \text{ differ} \\ 0 & \text{if } e \notin C. \end{cases} \quad (3)$$

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- It is easy to see that $Q\chi_C = 0$ (for each vertex of the cycle we deliver and take out the same amount of product)
- The fact that the cycle space is spanned by the signed indicator functions of cycles is based on Linear Algebra and some results on spanning trees. Namely, we prove that the number of linearly independent signed indicator functions of cycles is equal to the dimension of the kernel of Q.

- The representation $\mathrm{TC}(G) = \ell_{1,d}(E(G)))/Z(G)$ implies much more than the polyhedrality of $\mathrm{TC}(G)$ and the fact that the dimension of the needed $\ell_{1,d}(E(G)))$ does not exceed $\frac{|V(G)|(|V(G)|-1)}{2}$.

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- This study continued in [DKO21] and [DKO24].

What about geometry of cycle spaces?

- In connection with the discussed representation $\mathrm{TC}(G) = \ell_{1,d}(E(G))/Z(G)$, it would be very important to study the Banach space structure of $Z(G)$ and its "location" in $\ell_{1,d}(E(G))$.

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- The results of [DKO20, DKO21, DKO24] seem to go in a different direction.
- The only result which provide some Banach-space-theoretical consequences of the desired type is the following result of P. Erdős, L. Pósa [EP62]: The number of edge disjoint cycles in a cycle space of dimension d is $\Omega\left(\frac{d}{\log d}\right)$ and it is the best possible estimate. Note that disjoint cycles in $\ell_{1,d}(E(G))$ span an isometric to ℓ_1^k 1-complemented subspace.

Comments

- The representation $\mathrm{TC}(G) = \ell_{1,d}(E(G)))/Z(G)$ implies that if a transportation plan for μ (described as an element of $\ell_{1,d}(E(G))))$ is not optimal, then we can add to it an element of the cycle space to get a cheaper plan.

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- Many years ago (1939), Tolstoi noticed that this statement can be strengthened to the following **Tolstoi type theorem**: if a transportation plan for μ (described as an element of $\ell_{1,d}(E(G)))$) is not optimal, then we can add to it a multiple of the signed indicator function of a cycle to get a cheaper plan.

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- This Tolstoi was an economist, and he never proved this theorem. Different people published proofs of some versions. It seems that the most suitable for Banach space theory Tolstoi type theorem was proved in [OO22].

Geometric meaning of Tolstoi type theorem

- Recall the statement: if a transportation plan for μ (described as an element of $\ell_{1,d}(E(G))$)) is not optimal, then we can subtract from it a multiple of the signed indicator function of a cycle to get a cheaper plan.

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- We describe the plan as a set of edges with directions and quantities of the product which are to be moved along them.
- The geometric meaning is that the plan is not optimal if and only if there exists a directed cycle, for which the total length (weight) of edges which follow the transportation plan is more than half of the total length.

Optional: Another way of getting $\mathrm{TC}(X)$ as a quotient

- **Definition:** A point x in a nonempty convex set A of a vector space is called *extreme* if whenever $x = \alpha u + (1-\alpha)v$ with $\alpha \in (0,1)$ and with $u, v \in A$, then $x = u = v$.

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- **Minkowski theorem:** A compact convex set in $\mathbb{R}^n$ is a closed convex hull of its extreme points.
- **Observation** If a finite-dimensional normed space Y is such that its unit ball has exactly $2m$ extreme points, then Y is a quotient of ℓ_1^m and Y is not a quotient of ℓ_1^k for $k < m$.

- **Proposition:** If X is a finite metric space, extreme points of the unit ball of $\mathrm{TC}(X)$ are transportation problems of the form

$$m_{u,v} := \frac{1}{d(u,v)}(\mathbf{1}_u - \mathbf{1}_v), \tag{4}$$

where $u, v \in X$ are such that uv is an edge in the canonical graph of (X, d), that is, there are no points $z \in X$, $z \notin \{u, v\}$ such that $d(u,z) + d(z,v) = d(u,v)$. (This proposition implies the desired estimate for the number of extreme points from above.)

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- **Remark (Extreme points of $\mathrm{TC}(X)$ for infinite X)** The statement of Proposition remains true for infinite metric spaces (Aliaga, Pernecká, Smith [APS26+]). This result is much more difficult than its finite version.

Another geometric result on $\mathrm{TC}(M)$

- **Theorem (Khan, Mim, M.O., 2020):** If a metric space M contains $2n$ elements, then $\mathrm{TC}(M)$ contains a 1-complemented subspace isometric to ℓ_1^n. (1-*complemented* means that there exists a linear map from $\mathrm{TC}(M)$ onto this subspace which does not increase norms.)
  If the space M is such that triangle inequalities for all distinct triples in M are strict, then $\mathrm{TC}(M)$ does not contain a subspace isometric to ℓ_1^{n+1}.

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  If the space M is such that triangle inequalities for all distinct triples in M are strict, then $\mathrm{TC}(M)$ does not contain a subspace isometric to ℓ_1^{n+1}.
- The most difficult part of this result is that the subspace is 1-complemented. We shall not prove it because it uses a rather lengthy argument based on the analysis of the Edmonds algorithm for finding a minimum weight perfect matching.
- I should note that an earlier result in [DKO20] proving that there exists a 2-complemented 2-isomorphic ℓ_1^n has a simpler proof.

Presence of isometric ℓ_1^n

- Consider M as a weighted graph K_{2n}. The weight of an edge is the distance between its ends. Pick in this graph a minimum weight perfect matching, that is partition M into a set of pairs $\{u_i, v_i\}_{i=1}^n$ such that the sum $\sum_{i=1}^n d(u_i, v_i)$ is the least possible.

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- We claim that $f_i := \dfrac{\mathbf{1}_{u_i} - \mathbf{1}_{v_i}}{d(u_i, v_i)}$, $i = 1, \dots, n$ satisfy $\|\sum_{i=1}^n a_i f_i\|_{\mathrm{TC}} = \sum_{i=1}^n |a_i|$, so these vectors span a subspace isometric to ℓ_1^n.

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- We need estimate the norm from below only (see next slide).

- Assume the contrary, then

$$\sum_{i=1}^{n} \frac{a_i}{d(u_i, v_i)} (\mathbf{1}_{u_i} - \mathbf{1}_{v_i})$$

is not an optimal transportation plan for $\sum_{i=1}^{n} a_i f_i$.

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- This implies that $\{u_i, v_i\}_{i=1}^{n}$ is not a minimum weight perfect matching (sketch), we get a contradiction.

No ℓ_1^{n+1} if all triangles are "strict"

- **Enough:** If all triangle inequalities in M are strict and $\{f_i\}_{i=1}^{n+1}$ are isometrically equivalent to the unit vector basis of ℓ_1^{n+1}, then $\{f_i\}$ have disjoint supports.

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- Proof. Assume the contrary, let $v \in M$ be in the supports of both f_i and f_j, $i \neq j$. Without loss of generality we assume that $f_i(v) > 0$ and $f_j(v) < 0$.

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- Let $\gamma = \min\{\alpha, \beta\}$. Now we combine the optimal plans for f_i and f_j with the following exception: we move γ units of the product directly from w to u. Since, by our assumption, $d(w, u) < d(w, v) + d(v, u)$, the cost of the obtained plan is $< \|f_i\|_{\mathrm{TC}} + \|f_j\|_{\mathrm{TC}}$.

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- **Theorem:** For every n-dimensional Banach space B and any $\varepsilon > 0$, there is $N = N(n, \varepsilon)$ and a metric space X with $|X| = N$ such that there is linear embedding $T : B \to \mathrm{TC}(X)$ satisfying $\forall x \in B \quad \|x\| \le \|Tx\| \le (1 + \varepsilon)\|x\|$. (*)

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- (2) Deep result of Bourgain [Bou87]: For every n and $\varepsilon > 0$, there is $\delta > 0$ such that for every n-dimensional Banach space X and for any Banach space Y we have: if there is an isometric embedding of a δ-net N_δ of the unit ball of B into a Banach space Y, then there is an embedding $T : B \to Y$ satisfying (*)

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- To complete the proof, we denote by $N(n, \varepsilon)$ the estimate for $|N_\delta|$, and let $X = N_\delta$.

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- EMD in high-dimensional spaces is often used to compare documents and even two-dimensional color images.

Complexity of TC and EMD computations

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- Let $\{u_i\}_{i=1}^{s} = \{v : \ \mu(\{v\}) > 0\}$ and $\{v_j\}_{j=1}^{t}$ $= \{v : \ \mu(\{v\}) < 0\}$. We consider only transportation plans of the form $\sum_{i=1}^{s}\sum_{j=1}^{t} p_{ij}(\mathbf{1}_{u_i} - \mathbf{1}_{v_j})$, where by p_{ij} $(p_{ij} \geq 0)$ we denote the amount which we plan to deliver from u_i to v_j.

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- So, we need to solve this linear programming problem.

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- Charikar [Cha02] and Indyk-Thaper [IT03] suggested to do the search using low-distortion embeddings into ℓ_1^d (of a suitable dimension), where fast nearest neighbor algorithms are known.
- It makes solutions approximate, which is OK for many applications, especially when it is known that the exact solution takes much more time.

The problem of L_1-embeddings of $\mathrm{TC}(X)$

- The L_1-**distortion** $c_1(Z)$ of a metric space (Z, d) is the **infimal** D for which
$$d(x,y) \le \|f(x) - f(y)\|_1 \le Dd(x,y)$$
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- **Consequence.** This activated the study of embeddings:
$$\mathrm{TC}(X),\ \mathrm{EMD}(X) \hookrightarrow \ell_1^d$$

What is known on L_1-distortions

- We shall use the following general fact for finite X [NS07]: $c_1(\mathrm{TC}(X)) = c_1(\mathrm{EMD}(X))$, and for $\mathrm{TC}(X)$ we may restrict our attention to linear maps f. **Note:** This result is proved in [NS07] is a slightly weaker form and only for planar grids. Modifications needed to a general case were added in [GO26].

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- **Approximation by trees** ([AKPW95], [Bar96, Bar98], [Cha02], [FRT04])
- A key method for L_1-embeddability of $\mathrm{TC}(X)$ is to approximate the metric space (X, d) by random dominating tree metrics.

- **Definition:** We say that a finite metric space (X, d) is *embeds into a convex combination of dominating tree metrics with distortion* D or *is probabilistically (stochastically) approximated by dominating tree metrics with distortion* D if there is a finite collection of weighted trees $\{(T_i, d_i)\}_{i=1}^{m}$, such that

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This direction started with Karp's observation

- The first nontrivial example of this type (due to Karp) is a cycle C_n (with an arbitrarily large n) with vertices $\{1, \dots, n\}$ and its graph distance. It can be approximated by the trees $\{T_i\}_{i=1}^n$, each of which is obtained by deleting one edge in the cycle. Below you see C_6 and its T_2.

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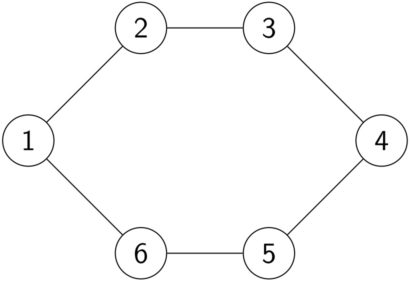

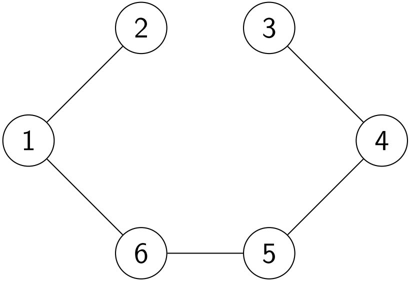

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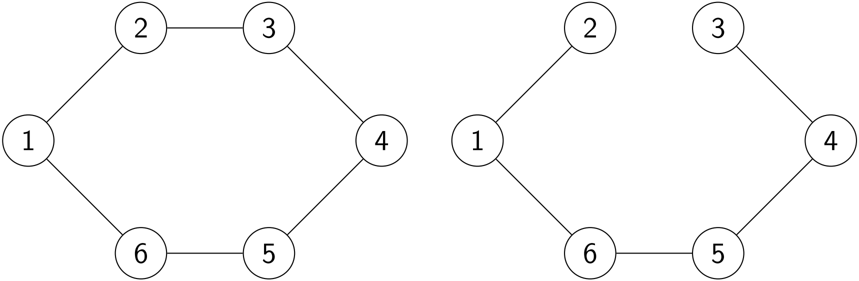

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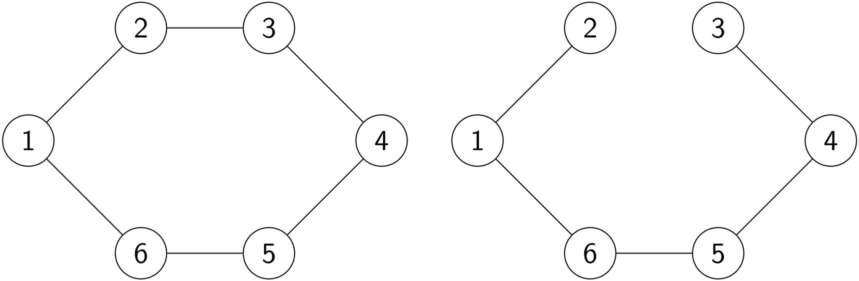

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- And easy that $\forall u, v \in C_n \quad \sum_{i=1}^n \frac{1}{n} d_{T_i}(u, v) \leq 2d_{C_n}(u, v)$.

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- One can show that constant degree expanders provide the lower bound $c \log n$ in both cases. For spanning trees there is still a gap between upper and lower bounds.

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- **Theorem:** If a finite metric space X is probabilistically approximated by dominating tree metrics with distortion $\le D$, then $\mathrm{TC}(X)$ admits a linear embedding into L_1 whose distortion does not exceed D.
- Charikar [Cha02] was the first to observe this result (its proof is rather straightforward). Since Charikar did not state and explain it clearly, several other people proved and published this theorem somewhat later.

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- Let $F : \mathrm{TC}(X, d) \to (\oplus_{i=1}^m \mathrm{TC}(T_i, d_i))_1$ be given by $F\mu = \{\alpha_i \mu_i\}_{i=1}^m$. Earlier exercise: each $\mathrm{TC}(T_i, d_i)$, and thus the sum $(\oplus_{i=1}^m \mathrm{TC}(T_i, d_i))_1$, is isometric to ℓ_1^d of the corresponding dimension. Verification of the inequality $\|\mu\| \leq \|F\mu\| \leq D\|\mu\|$ is straightforward (summation order).

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- It became important to know: what is the order of $c_1(X)$ for these spaces?

- Bourgain [Bou86] for Hamming cubes. The most difficult. It is based on presence of high-dimensional ℓ_∞^k in $\mathrm{TC}(X)$. Could be applicable to other spaces listed below, but I would expect that these proofs would be harder than the existing.

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- Gartland-M. O.-Rabani-Young [GORY26] for high-dimensional grids. Based on Poincaré inequalities for random measures on grids. Could be transferred to other discrete structures by people who could understand how to handle random measures there.

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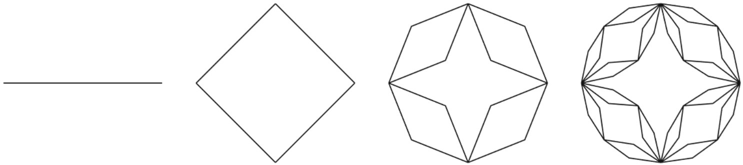
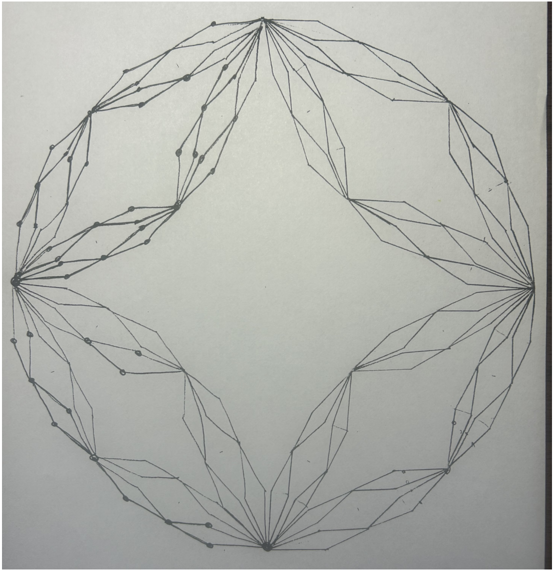
- 

D_0 D_1 D_2 D_3

Figure: The first four diamond graphs

Diamond D_4



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- The following things are the main difficulties: (1) Guess the right measures $\{\tau_\alpha\}_\alpha$, (2) Prove the statement of the previous paragraph.
- Experience showed us that a useful collection of measures can be built using "rooted trees" of measures.

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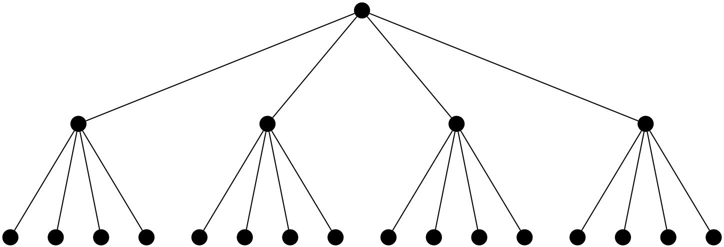

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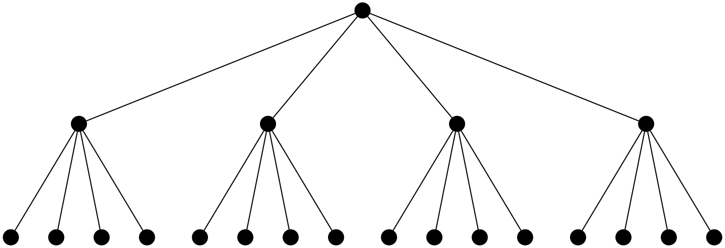
- 
  On this figure you see a 4-branching tree of depth 2. The top vertex is a *root*.
- For D_n, we need a tree of depth $n-1$, we denote it T. We introduce *level* k $(0 \le k \le n-1)$ of T, denoted T_k, as the set of vertices at distance k from the root. So T_0 is consists of the root only.

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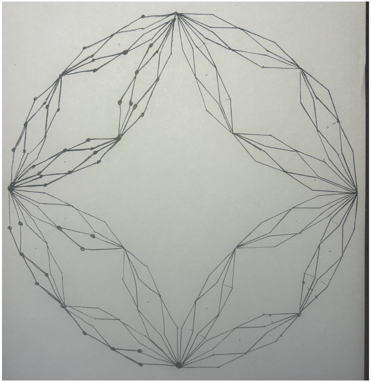
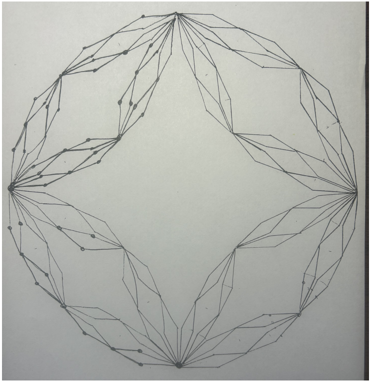


- **Terminology for diamonds:** To describe the measures, we introduce some terminology for diamonds.
- D_n is its own *subdiamond of level* 0. It has 4 *subdiamonds of level* 1 (they are isomorphic to D_{n-1}), and 16 *subdiamonds of level* 2 (isomorphic to D_{n-2}), and so on.



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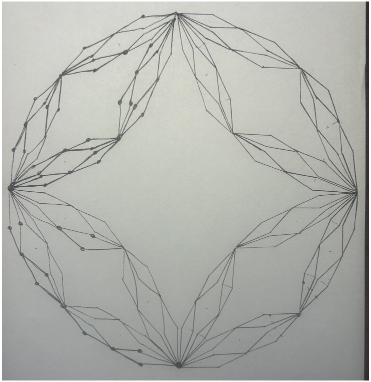
- For each of the subdiamonds we define the corresponding midpoints, top and bottom.
- D_4 

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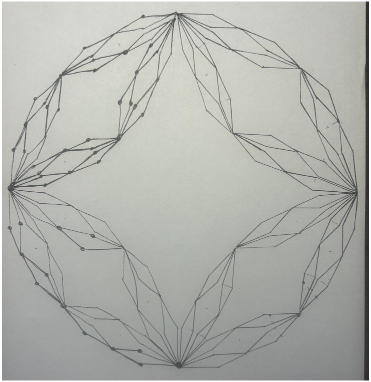
- D_4 

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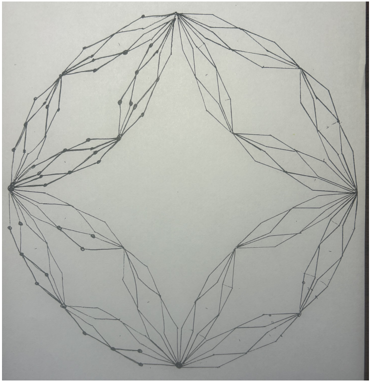
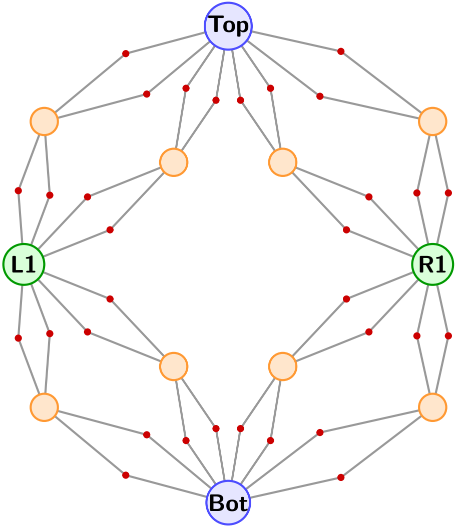
- D_4 
- In each of the 4^k subdiamonds of level k ($k \in \{0\dots,n-1\}$) we introduce a measure supported on the pair of midpoints of the subdiamond, with the masses $\pm 2^{n-k}\cdot 4^{-n}$, so that $\mu_t(D_n)=0$ and $\|\mu_t\|_{\mathrm{TC}}=4^{-k}$ for each $t\in T_k$.

Supports of measures: Level 0 - green, Level 1 - yellow, Level 2 - red



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- One of the important conditions for the tree of measures: the cost of the sum of measures of each level remains equal 1 after multiplying measures μ_t by ± 1 arbitrarily.

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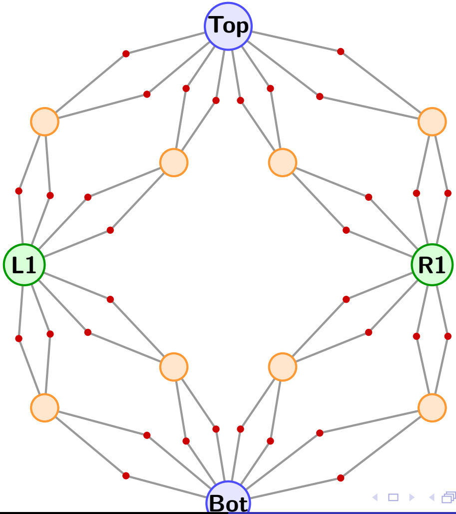

The inequality implying the main theorem

- We do not need all possible ± 1. The following set of signs is sufficient: Let T_k be the set of measures of level k. Let $\{\theta_s(t)\}_{s,t\in T_k}$ be any orthogonal matrix with entries ± 1. A sufficient set of signs is $\{\sum_{t\in T_k}\theta_s(t)\mu_t\}_{s\in T_k}$ (Hadamard).

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- We prove that choosing μ_t as above, we can achieve, for some absolute constant C,

$$\sum_{k=0}^{n-1}\left(\frac{1}{|T_k|}\sum_{s\in T_k}\left\|f\left(\sum_{t\in T_k}\theta_s(t)\mu_t\right)\right\|_{\ell_1^d}^2\right)^{1/2} \qquad (*)$$
$$\le \frac{C}{|E(D_n)|}\sum_{e\in E(D_n)}\|f(\mathbf{1}_{e^+}-\mathbf{1}_{e^-})\|_{\ell_1^d}.$$

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- As for the right-hand side. We use the fact that for each edge e, we have $\|\mathbf{1}_{e^+}-\mathbf{1}_{e^-}\|_{\mathrm{TC}}=1$. Therefore the right-hand side does not exceed $C\|f\|$, this completes the proof.

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- **Step 2:** Proof of the Sobolev inequality (for the defined above measures μ_t).

Step 1

- $$\left(\sum_{k=0}^{n-1}\left(\frac{1}{|T_k|}\sum_{s\in T_k}\left\|f\left(\sum_{t\in T_k}\theta_s(t)\mu_t\right)\right\|_{\ell_1^d}^2\right)\right)^{1/2}$$
$$\stackrel{\text{def}\ \|\cdot\|_1}{=}\left(\sum_{k=0}^{n-1}\left(\frac{1}{|T_k|}\sum_{s\in T_k}\left(\sum_{j=1}^{d}\left|f_j\left(\sum_{t\in T_k}\theta_s(t)\mu_t\right)\right|\right)^2\right)\right)^{1/2}$$
$$\stackrel{\text{triangle}}{\leq}\sum_{j=1}^{d}\left(\sum_{k=0}^{n-1}\left(\frac{1}{|T_k|}\sum_{s\in T_k}\left|f_j\left(\sum_{t\in T_k}\theta_s(t)\mu_t\right)\right|^2\right)\right)^{1/2}$$
$$\stackrel{\text{Parseval \& linearity}}{=}\sum_{j=1}^{d}\left(\sum_{k=0}^{n-1}\left(\sum_{t\in T_k}|f_j(\mu_t)|^2\right)\right)^{1/2}$$

Moving to Step 2

- So we got
$$\leq \sum_{j=1}^{d} \sum_{k=0}^{n-1} \left(\sum_{t \in T_k} |f_j(\mu_t)|^2 \right)^{1/2}$$
in the rightmost side.

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- So, it remains to prove the following Sobolev inequality
$$\sum_{k=0}^{n-1} \left(\sum_{t \in T_k} |f_j(\mu_t)|^2 \right)^{1/2} \le \frac{C}{|E(D_n)|} \sum_{e \in E(D_n)} |f_j(\mathbf{1}_{e^+} - \mathbf{1}_{e^-})|$$

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- The reduction is a rather straightforward consequence of the proved by M. O.[Ost05] characterization of extreme points in unit balls of Sobolev spaces on graphs.
- After that, it remains to prove that there is an absolute constant C such that for each simply connected subset A of $V(D_n)$ the following inequality holds (we use the integral for $\mathbf{1}_A(\mu_t)$).

$$\sum_{k=0}^{n-1}\left(\sum_{t\in T_k}\left|\int \mathbf{1}_A d\mu_t\right|^2\right)^{\frac{1}{2}} \le \frac{C}{|E(D_n)|}\sum_{(u,v)\in E(D_n)}|\mathbf{1}_A(v)-\mathbf{1}_A(u)|.$$

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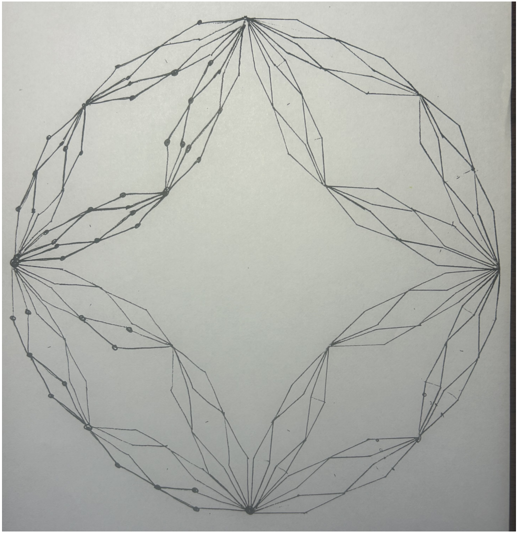
- We claim that in such a case A splits at most 2^k measures in T_k, $1 \leq k \leq n-1$.

Ways to split the measure in T_0



▶

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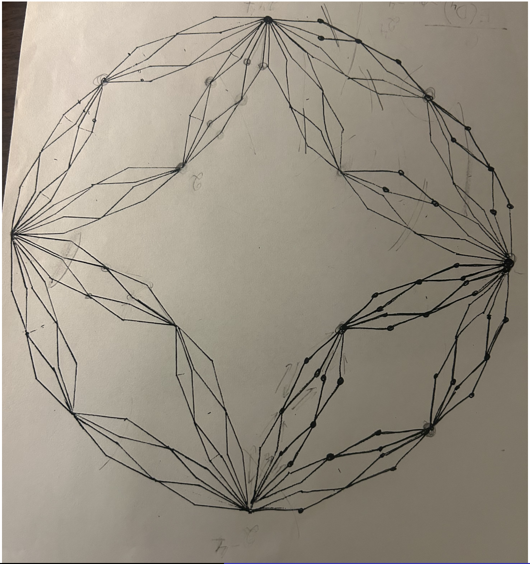
- A splits 2 subdiamonds of level 1 (and is simply connected) $\Rightarrow$ A splits 4 subdiamonds of level 2 $\Rightarrow$ splits at most 4 measures of level 2.
- So on. It proves the statement repeated above.

Proving the statement on measures splitting $V(D_1)$

- 

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- It remains to observe that the LHS can be transformed to $(2^{-n})(1+2^{-1/2}+\cdots+2^{-n/2})\le\frac{\sqrt{2}}{\sqrt{2}-1}\cdot 2^{-n}$. The inequality follows.

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- As above, the inequality can be proved summing the geometric series.

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