Lipschitz quotients and the curve-flat index

with J.K. Kaasik

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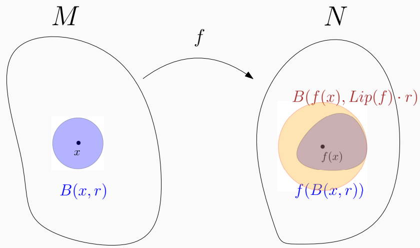
Lipschitz quotients

Definition

A map $f\colon M \to N$ between metric spaces is *Lipschitz* with Lipschitz constant $\mathrm{Lip}(f)$ if

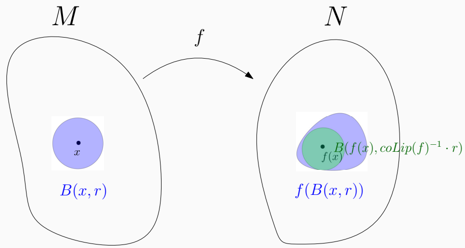
$$f(B(x,r)) \subset B(f(x), \mathrm{Lip}(f)\cdot r)$$



Lipschitz quotients

Definition
A map $f\colon M \to N$ between metric spaces is *co-Lipschitz* with Lipschitz constant $\text{co-Lip}(f)$ if

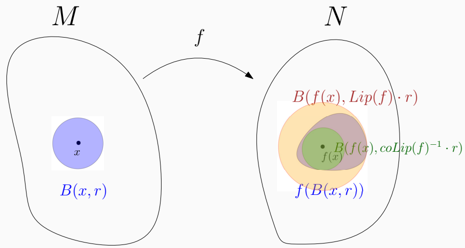
$$f(B(x,r)) \supset B(f(x), \text{co-Lip}(f)^{-1} \cdot r)$$



Lipschitz quotients

Definition
A map $f\colon M \to N$ between metric spaces is a *Lipschitz quotient* with constant C if $\mathrm{Lip}(f) \cdot \text{co-Lip}(f) \leq C$

$$B(f(x), \text{co-Lip}(f)^{-1} \cdot r) \subset f(B(x,r)) \subset B(f(x), \mathrm{Lip}(f) \cdot r)$$



Examples

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Examples of Lipschitz quotients

- Linear quotients (Open mapping theorem)
- In particular, linear projections
- Surjective Lipschitz maps onto uniformly discrete spaces.

Universality results

Definition

Given a family of metric spaces $\mathcal{F}$, we say that a space M is *universal for (C)-Lipschitz quotients onto $\mathcal{F}$* if every $N \in \mathcal{F}$ is a (C)-Lipschitz quotient of M.

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In particular, any Banach space containing ℓ_1 is universal for 1-Lipschitz quotients onto separable geodesic metric spaces.

Theorem (Kaasik, Q. '26)

There exists a separable complete metric space SUT such that every separable complete metric space is a 1-Lipschitz quotient of SUT.

The compact question

Question

Does there exist a *compact* metric space universal for Lipschitz quotients onto all compact metric spaces?

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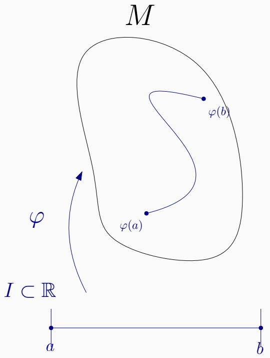
The answer is...

Theorem (Kaasik, Q. '26)

There is **no** universal compact metric space for Lipschitz quotients onto compact metric spaces.

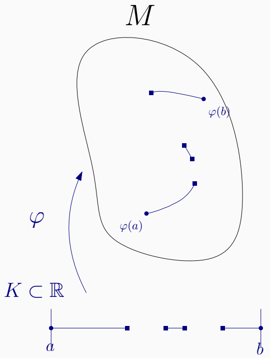
Definition

A ***Lipschitz curve*** is a bi-Lipschitz map $\varphi\colon I \to M$ where I is compact interval of $\mathbb{R}$.



Definition

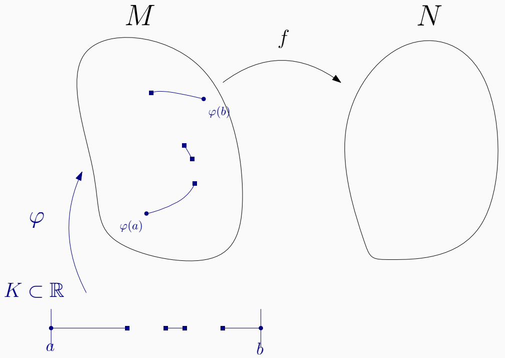
A *Lipschitz curve fragment* is a bi-Lipschitz map $\varphi\colon K \to M$ where K is compact subset of $\mathbb{R}$.



Lipschitz curves and curve-fragments

Remark

Lipschitz curve-fragments do not necessarily pass onto Lipschitz quotients. The bi-Lipschitz condition may break.



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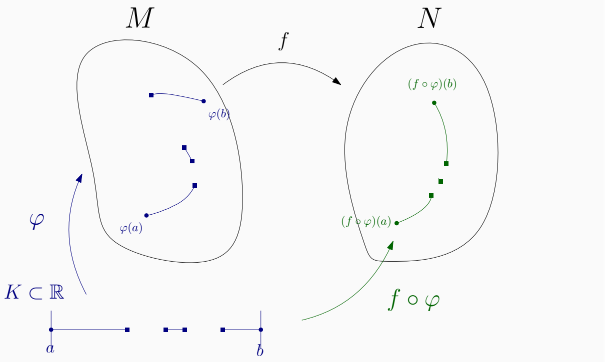
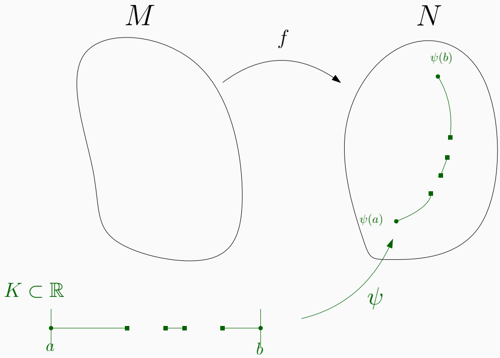


Proposition
For compact domains, Lipschitz curve-fragments in the image *do* induce curve-fragments in the domain.



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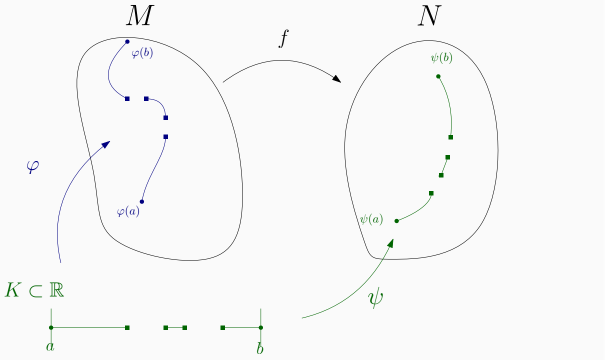
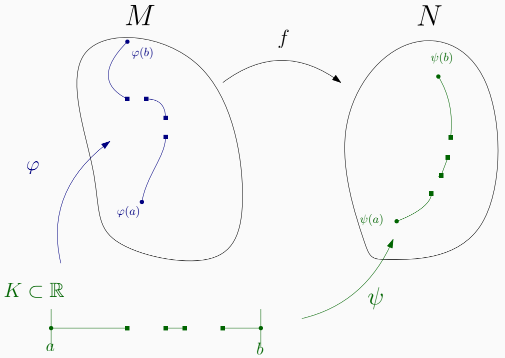


Lipschitz curves and curve-fragments

Informal corollary
If N is a Lipschitz quotient of a compact M, then N cannot have "many more" Lipschitz curve-fragments than M



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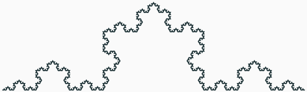
- Snowflakes: (M, d^α), $\alpha < 1$.



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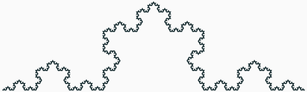

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Corollary (Kun, Maleva, Máthé, RME '06) (Kaasik, Q. '26)

A Lipschitz quotient of a purely 1-unrectifiable compact metric space is also purely 1-unrectifiable.

The curve-flat pseudometric

Definition (Aliaga, Gartland, Petitjean, Procházka, TAMS '22)

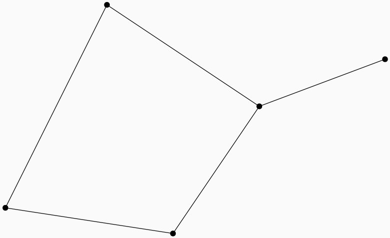
Given a metric space (M, d), the *curve-flat pseudo-metric* d_{cf} on M is defined by:

$$d_{cf}(x, y) := \inf(\lambda([\min K, \max K] \setminus K)),$$

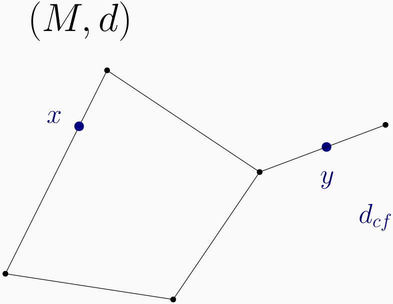
for all 1-Lipschitz curve-fragments $\varphi\colon K \to M$ with $\varphi(\min K) = x$, and $\varphi(\max K) = y$.

The curve-flat pseudometric

(M, d)



The curve-flat pseudometric



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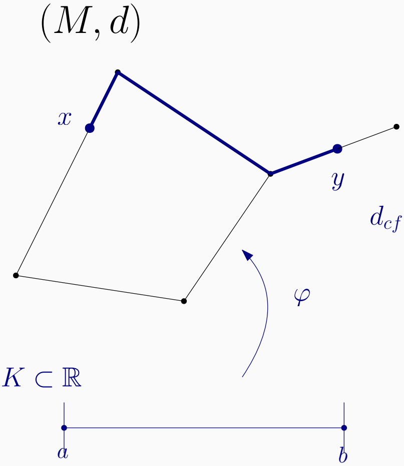

$$d_{cf}(x, y) = \inf(\lambda([\min K, \max K] \setminus K)$$

$$d_{cf}(x, y) = 0$$

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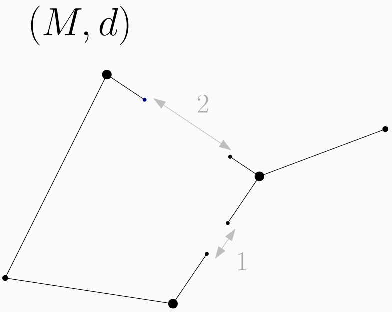

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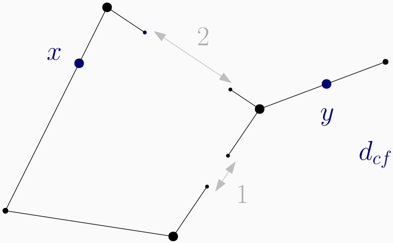

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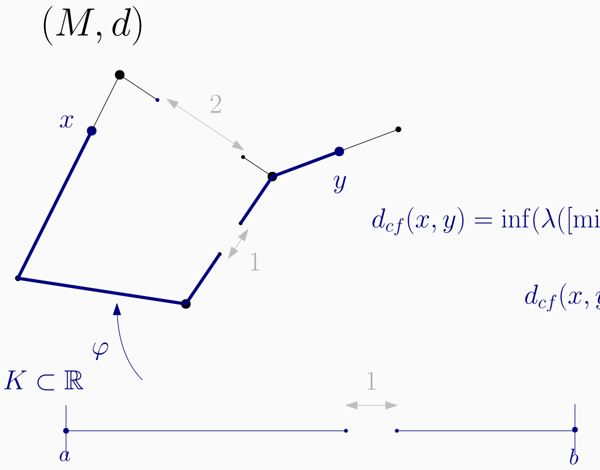
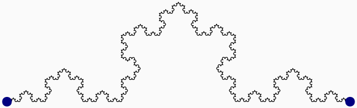


$$d_{cf}(x, y) = \inf(\lambda([\min K, \max K] \setminus K)$$

$$d_{cf}(x, y) = 1$$

The curve-flat pseudometric

$$([0,1], |\cdot|^{\alpha})$$



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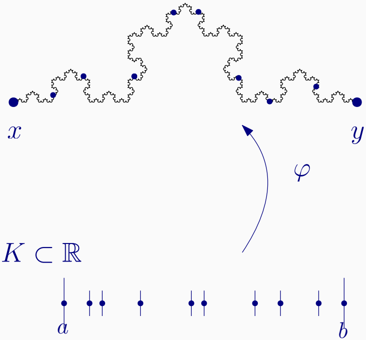

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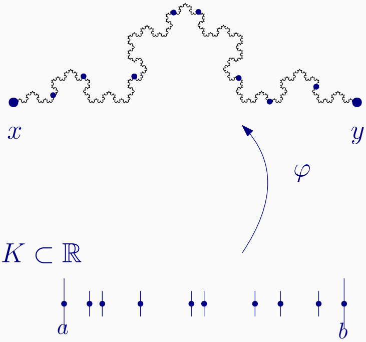

$$d_{cf}(x,y) = \inf(\lambda([\min K, \max K] \setminus K)$$

$$d_{cf}(x,y) = d(x,y)$$

$K \subset \mathbb{R}$

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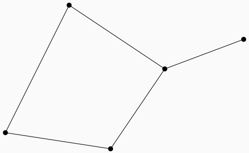
$$M_{cf} := (M/d_{cf}, d_{cf})$$

The curve-flat quotient

$M_1 \equiv$  $(M_1)_{cf} \equiv \quad \bullet$

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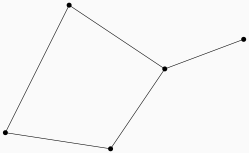
$M_1 \equiv$ 

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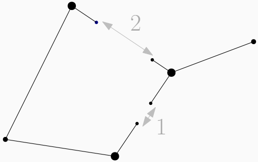
$M_2 \equiv$ 

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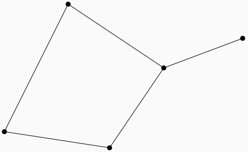
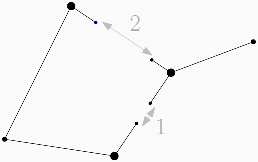
$M_1 \equiv$  $(M_1)_{cf} \equiv$ 

$M_2 \equiv$ 

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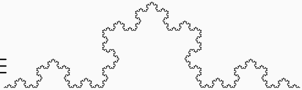
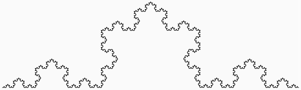
$M_3 \equiv$  $(M_3)_{cf} \equiv$ 

The curve-flat index

- $d_{cf} \leq d$
- $d_{cf} = d$ if and only if (M, d) is purely 1-unrectifiable.
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$$M = M_{cf}^0 \to M_{cf}^1 \to \cdots \to M_{cf}^\alpha \to \dots$$

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If M separable, there exists separable $\alpha < \omega_1$ such that M_{cf}^α is purely 1-unrectifiable.

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The least such α is $\alpha_{cf}(M)$, the *curve-flat index of M*.

Lipschitz quotients preserve cf index

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We "only" need to construct compact spaces $(M_\beta)_{\beta<\omega_1}$ with $\alpha_{cf}(M_\alpha) \geq \beta$.

A very general solution

Theorem (Kaasik, Q. '26)

Let M be a compact metric space, let $\alpha < \omega_1$ and let $\varepsilon > 0$. Then there exists a compact Y s.t.

$$Y^{\alpha}_{cf} \equiv_{(1+\varepsilon)} M$$

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This solves almost completely a recent question...

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There is no compact space universal for Lipschitz quotients onto all compact metric spaces.

The gapped graph construction

Goal:

We want a compact M such that $M_{cf} \equiv [0,1]$

$$([0,1], |\cdot|)$$

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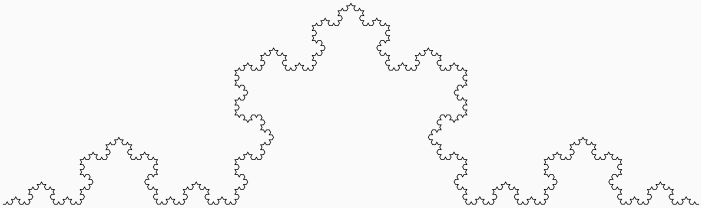

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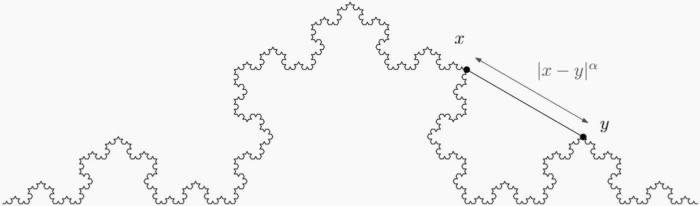

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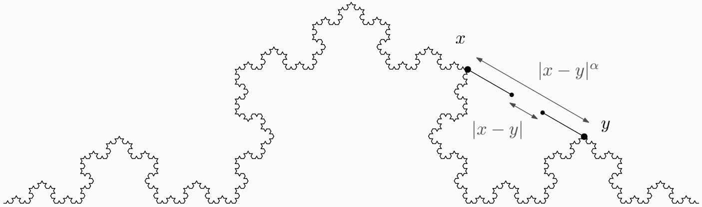

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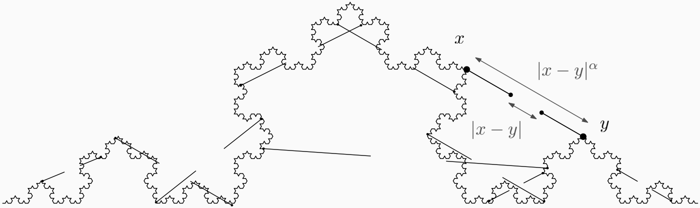

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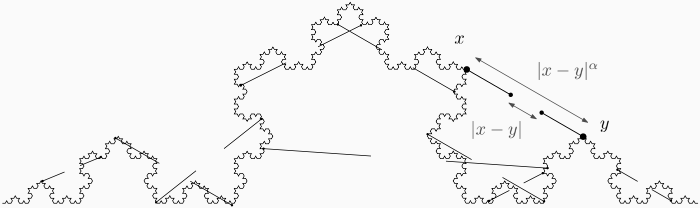
Then $M_{cf} = [0,1]$, but M is non-separable...



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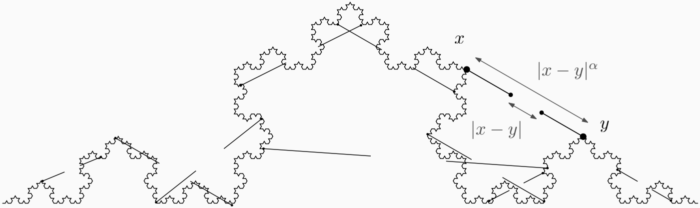

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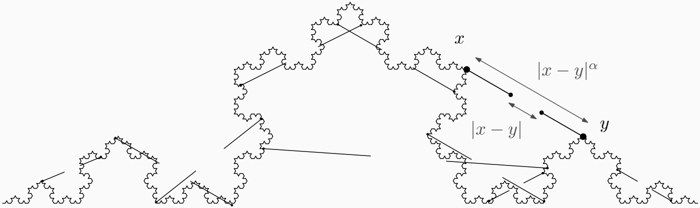
Then still $M_{cf} = [0,1]$, but M is not compact...



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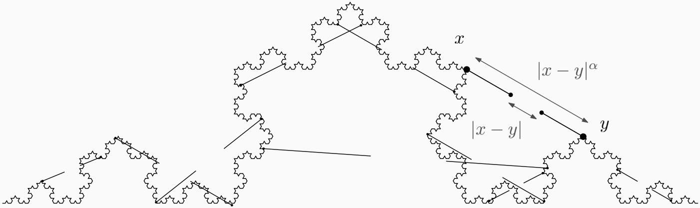
We want a compact M such that $M_{cf} \equiv [0,1]$

We add gapped segments for all $(x, y) \in F_n$,
with $F_n \subset [0,1]^2$ a specific family of pairs



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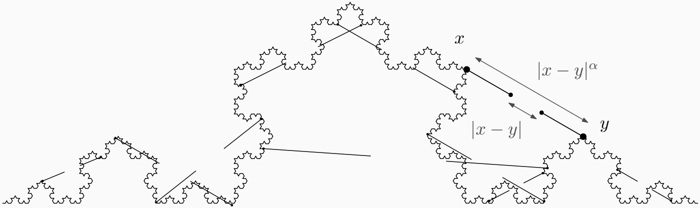
Then M is compact at last, but we must pay $\varepsilon > 0$:
M_{cf} is $(1+\varepsilon)$-isometric to $[0,1]$



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Open problems

Problem 1

Can we get high order compact spaces with extra properties? E.g:

- With finite Nagata dimension?
- As doubling spaces?
- As subsets of $\mathbb{R}^n$?

This would lead to stronger non-universality results.

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Problem 2

Some properties force small curve-flat index

- Compact purely 1-unrectifiable spaces ($\alpha_{cf}(M) = 0$)
- Compact subsets of $\mathbb{R}$ ($\alpha_{cf}(M) \leq 1$)

Do these families admit universal compact spaces for Lipschitz quotients?

Thank you for you attention!