A group action approach to the Daugavet property

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Daugavet property

Definition

A Banach space X has the **Daugavet property** if for every rank-one operator $T : X \to X$ we have

$$\|\mathrm{Id} + T\| = 1 + \|T\| .$$

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Definition

A Banach space X has the **alternative Daugavet property** if for every rank-one operator $T : X \to X$ we have

$$\max_{\theta \in S_{\mathbb{K}}} \|\mathrm{Id} + \theta T\| = 1 + \|T\| .$$

M. Martín and T. Oikhberg. An alternative Daugavet property. *J. Math. Anal. Appl.*, 294:158–180, 2004

Characterisation of the Daugavet property

Proposition (Daugavet property)

Let X be a Banach space. The following are equivalent.

1. $X \in \mathrm{DPr}$.
2. *For every* $\varepsilon > 0$, $x \in S_X$ *and slice* S *of* B_X, *there is* $y \in S$ *such that*
$$\|x + y\| > 2 - \varepsilon.$$
3. *For every* $\varepsilon > 0$, $x \in S_X$ *and slice* S *of* B_X, *there is* $y \in S$ *such that*
$$\|x - y\| > 2 - \varepsilon.$$
4. *For every* $\varepsilon > 0$ *and* $x \in S_X$, *we have*
$$\overline{\mathrm{co}}(Q(x, \varepsilon)) = B_X,$$
where
$$Q(x, \varepsilon) = \{y \in B_X : \|x + y\| > 2 - \varepsilon\}.$$
5. *Many more...*

Proposition (Alternative Daugavet property)

Let X be a complex Banach space and let $S_{\mathbb{K}} = \{\omega \in \mathbb{K} : |\omega| = 1\}$. The following are equivalent.

1. *X has the alternative Daugavet property.*
2. *For every $\varepsilon > 0$, $x \in S_X$ and slice S of B_X, there are $y \in S$ and $\omega \in S_{\mathbb{K}}$ such that*
$$\|x + \omega y\| > 2 - \varepsilon.$$
3. *For every $\varepsilon > 0$, $x \in S_X$ and slice S of B_X, there are $y \in S$ and $\omega \in S_{\mathbb{K}}$ such that*
$$\|x - \omega y\| > 2 - \varepsilon.$$
4. *For every $\varepsilon > 0$ and $x \in S_X$, we have*
$$\overline{\mathrm{co}}(Q_{S_{\mathbb{K}}}(x, \varepsilon)) = B_X,$$
where
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The (a)DP for $L^1(\mu, X)$ and $C(K)$

Proposition

Let X be a Banach space and let (Ω, Σ, μ) be a measure space. Then $L^1(\mu, X)$ has the (a)DP if and only if μ is atomless or X has the (a)DP.

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Proposition

Let X be a Banach space and let K be a compact Hausdorff space. Then $C(K, X)$ has the (a)DP if and only if K is perfect or X has the (a)DP.

The (a)DP and the RNP

Proposition

If a Banach space X has the Daugavet property, then X does not have the Radon-Nikodým property.

Proposition

If a real Banach space X has the alternative Daugavet property and the Radon-Nikodým property, then $n(X) = 1$.

The annoyance

They are similar...

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The proof is a straightforward adaptation of those given in [24, Lemma 2.1] and [45, Corollary 2.3], so we omit it. Using this result, we can prove an analogue of [24, Theorem 2.3].

Group actions

Let G be a group and X be a Banach space.

- A **group action** of G on X is a map $\alpha : G \times X \to X$ such that for all $g, h \in G$ and $x \in X$:
  - $\alpha(e, x) = x$, where e is the identity element of G,
  - $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$.

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- The action is **by linear isometries** if for each $g \in G$, the map $x \mapsto \alpha(g, x)$ is a linear isometry on X.
- For brevity, we often write $g \cdot x$ or gx instead of $\alpha(g, x)$.

We then say that X is a G-**Banach space**.

Notation

- $G \cdot A$: G-orbit of a set $A \subset X$, i.e.,

$$G \cdot A = \{g \cdot a : g \in G, a \in A\};$$

- A is G-invariant if $G \cdot A = A$;
- $\overline{\mathrm{co}}_G(A)$: closed convex hull of the G-orbit of a set $A \subset X$, i.e.,

$$\overline{\mathrm{co}}_G(A) = \overline{\mathrm{co}}\, G \cdot A.$$

The G-DP

A Banach space X has the **Daugavet property** if for every rank-one operator $T : X \to X$ we have

$$\max_{\theta \in \{1\}} \|\mathrm{Id} + \theta T\| = 1 + \|T\|.$$

A Banach space X has the **alternative Daugavet property** if for every rank-one operator $T : X \to X$ we have

$$\max_{\theta \in S_{\mathbb{K}}} \|\mathrm{Id} + \theta T\| = 1 + \|T\|.$$

The G-DP

Definition

Let X be a G-Banach space. We say that X has the *G-Daugavet property* (G-DPr, for short) if every rank-one operator $T : X \to X$ satisfies the *G-Daugavet equation*

$$\sup_{g \in G} \|\mathrm{Id} + g \circ T\| = 1 + \|T\|.$$

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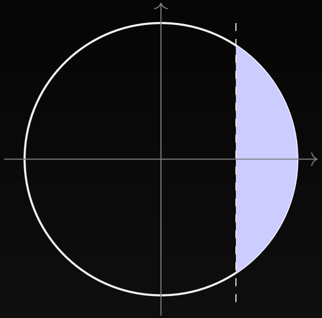
Definition

Let X be a G-Banach space, $\varepsilon > 0$ and $x^* \in S_{X^*}$. A *G-slice of B_X* is a set of the form

$$\begin{aligned} S_G(B_X, x^*, \varepsilon) &= \{y \in B_X : (\exists g \in G)\ \operatorname{Re} x^*(gy) > 1 - \varepsilon\} \\ &= \bigcup_{g \in G} gS(B_X, x^*, \varepsilon). \end{aligned}$$

$X = \mathbb{R}^2$, $x^* = (1,0)$,
$G = \{\text{rotations by } \pi/2\}$



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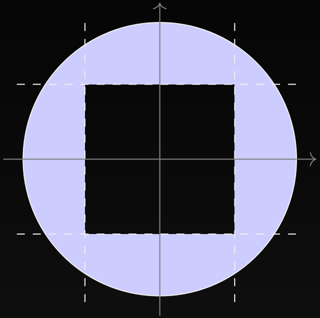

$S_G(B_X, x^*, \varepsilon)$

Basic characterisation of the G-DP

Proposition

Let X be a G-Banach space. The following are equivalent.

1. $X \in G\text{-}DPr$.
2. *For every* $\varepsilon > 0$, $x \in S_X$ *and* G*-slice* S_G *of* B_X, *there is* $y \in S_G$ *such that* $\|x + y\| > 2 - \varepsilon$.
3. *For every* $\varepsilon > 0$, $x \in S_X$ *and* G*-slice* S_G *of* B_X, *there is* $y \in S_G$ *such that* $\|x - y\| > 2 - \varepsilon$.
4. *For every* $\varepsilon > 0$ *and* $x \in S_X$, *we have that* $\overline{\mathrm{co}}_G(Q(x, \varepsilon)) = B_X$, *where* $Q(x, \varepsilon) = \{y \in B_X : \|x + y\| > 2 - \varepsilon\}$.
5. *Every* G*-slice* S_G *of* B_X *is such that* $S_G \cap Q(x, \varepsilon) \neq \emptyset$ *for every* $\varepsilon > 0$ *and every* $x \in S_X$.
6. *For every* $x \in S_X$, G*-slice* S_G *of* B_X *and* $\delta > 0$, *there is a further* G*-slice* $\tilde{S}_G$ *of* B_X *such that* $\tilde{S}_G \subseteq S_G$ *and* $z \in Q_G(x, \delta)$ *for every* $z \in \tilde{S}_G$.
7. *For every* $x \in S_X$ *and* $\varepsilon > 0$, *we have that* $\overline{\mathrm{co}}_G(S_X \setminus (x + (2 - \varepsilon)B_X)) = B_X$.

The G-numerical index

Definition

Let X be a G-Banach space. For $T \in \mathcal{L}(X)$, define the G-**numerical radius**

$$v_G(T) = \sup\{\operatorname{Re} x^*(T(gx)) : x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1,\ g \in G\}.$$

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$$n_G(X) = \inf\{v_G(T) : T \in \mathcal{L}(X),\ \|T\| = 1\}.$$

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If $G = S_{\mathbb{K}}$ acts by scalar multiplication, then $v_G(T) = v(T)$ and $n_G(X) = n(X)$.

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If $G = S_{\mathbb{K}}$ acts by scalar multiplication, then $v_G(T) = v(T)$ and $n_G(X) = n(X)$.

If $G = \{\mathrm{Id}\}$, then $v_G(-\mathrm{Id}) = -1$ and $n_G(X) = -1$.

The G-DP and the RNP

Theorem

Let X be a G-Banach space. If X has both the Radon-Nikodým property and the G-DPr, then

$$n_G(X) = 1.$$

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Theorem

Let X be a G-Banach space. If X has both the Radon-Nikodým property and the G-DPr, then

$$n_G(X) = 1.$$

This recovers the classical statements:

- if X has the RNP and the aDPr, then $n(X) = 1$;
- if X has the DPr, then X does not have the RNP.

LUR spaces and transitive actions

Recall that an action is **almost transitive** if

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A Banach space X is **LUR** if

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Theorem

Let X be an LUR G-Banach space. Then X has the G-DPr if and only if the action of G on S_X is almost transitive.

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Theorem

Let X be an LUR G-Banach space. Then X has the G-DPr if and only if the action of G on S_X is almost transitive.

Corollary

Let $1 < p < \infty$.

- *$L^p[0,1]$ has the $\mathrm{Iso}(L^p[0,1])$-Daugavet property.*
- *$L^p[0,1]$ fails the $\mathrm{Iso}^+(L^p[0,1])$-Daugavet property.*
- *If $p \neq 2$ and $G \leq \mathrm{Iso}(\ell_p)$, then ℓ_p fails the G-Daugavet property.*

L^p and ℓ_p

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- L^p, $p \in (1, \infty)$ has the $\mathrm{Iso}(L^p)$-DP, but not the DP or the aDP;
- ℓ_p, $p \in (1, \infty) \setminus \{2\}$ does not have the G-DP for any G.

The constant α_G

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- $0 \leq \alpha_G(X) \leq 2$.
- $\alpha_{\{\mathrm{Id}\}}(X) = 0$.
- If $S_{\mathbb{K}} \subseteq G$, then $\alpha_G(X) = 2$.
- $\alpha_G(X) < 2$ means the action is uniformly "non-antipodal".

$\alpha_G < 2$: back to Daugavet-like behaviour

Theorem

Let X be a G-Banach space with the G-DPr and suppose that $\alpha_G(X) < 2$.

- *X does not have the RNP; in particular, X is not reflexive.*
- *If X has an unconditional basis, then X contains a copy of c_0.*

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