Bounds on injectivity of ℓ_∞/c_0

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Injectivity

Hahn-Banach theorem

Let Y be a Banach space over field $\mathbb{K}$ and $Z \subseteq Y$ its closed subspace. If $\Phi : Z \to \mathbb{K}$ is a bounded linear functional, then there exists $\bar{\Phi} : Y \to \mathbb{K}$, a bounded linear extension of Φ *(such that* $\|\bar{\Phi}\| = \|\Phi\|$*)*.

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- $\ell_\infty(\kappa)$ is injective
- spaces $C(K)$ for K extremally disconnected are injective
- spaces isomorphic to an injective space are also injective
- anything else? (we don't know)

Space ℓ_∞/c_0

- $f \sim g \iff f - g \in c_0$ for $f, g \in \ell_\infty$
- $[f] = \{g \in \ell_\infty : f \sim g\}$
- $\ell_\infty/c_0 = \{[f] : f \in \ell_\infty\}$ with the quotient norm $\|[f]\| = \inf\{\|g\| : g \in [f]\}$

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Observation

If $\mathcal{A} \subseteq \wp(\mathbb{N})$ is an almost disjoint family, then $Y_{\mathcal{A}} := \overline{\text{span}}\{[1_A] : A \in \mathcal{A}\}$ is an isometric copy of $c_0(|\mathcal{A}|)$ in ℓ_∞/c_0.

Non-injectivity of ℓ_∞/c_0

Theorem (Rosenthal)

Let X be a Banach space, Γ an infinite set and $T : \ell_\infty(\Gamma) \to X$ an operator such that $T|c_0(\Gamma)$ is an isomorphism. Then there is a set $\Gamma' \subseteq \Gamma$ with $|\Gamma'| = |\Gamma|$ such that $T|\ell_\infty(\Gamma')$ is an isomorphism.

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ℓ_∞/c_0 is not injective.

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Proof:

- there is an almost disjoint family $\mathcal{A} \subseteq \wp(\mathbb{N})$ of cardinality $\mathfrak{c}$
  $\implies \ell_\infty/c_0$ contains a subspace $Y_{\mathcal{A}}$ which is isomorphic to $c_0(\mathfrak{c})$

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- we have $S : c_0(\mathfrak{c}) \to \ell_\infty/c_0$, an isomorphic embedding onto $Y_\mathcal{A}$

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- we have $S : c_0(\mathfrak{c}) \to \ell_\infty/c_0$, an isomorphic embedding onto $Y_\mathcal{A}$
- if S extends to an operator $T : \ell_\infty(\mathfrak{c}) \to \ell_\infty/c_0$, then there is $\Gamma' \subseteq \mathfrak{c}$ with $|\Gamma'| = \mathfrak{c}$ such that $T|\ell_\infty(\Gamma')$ is an isomorphism

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- if S extends to an operator $T : \ell_\infty(\mathfrak{c}) \to \ell_\infty/c_0$, then there is $\Gamma' \subseteq \mathfrak{c}$ with $|\Gamma'| = \mathfrak{c}$ such that $T|\ell_\infty(\Gamma')$ is an isomorphism
- but $|\ell_\infty(\Gamma')| = 2^\mathfrak{c} > \mathfrak{c} = |\ell_\infty/c_0|$, a contradiction.

Separable injectivity of ℓ_∞/c_0

Theorem (Avilés, Cabello Sánchez, Castillo, González, Moreno)

Every separable subspace of ℓ_∞/c_0 is contained in a subspace isometrically isomorphic to ℓ_∞.

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Proof: $Z \subseteq Y$ – Banach spaces, $T : Z \to \ell_\infty/c_0$ – operator:

$$\begin{tikzcd} Y & & \ell_\infty/c_0 \\ Z \arrow[u, hook] \arrow[urr, "T"] & & \end{tikzcd}$$

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$$\begin{tikzcd}
Y & M \arrow[r, hook, "i"] & \ell_\infty/c_0 \\
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There is a subspace $M \subseteq \ell_\infty/c_0$ isometrically isomorphic to ℓ_∞ which contains $T[Z]$.

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There is a subspace $M \subseteq \ell_\infty/c_0$ isometrically isomorphic to ℓ_∞ which contains $T[Z]$. M is injective, so T can be extended.

Restricted injectivity

Let $\mathcal{C}$ be a class of Banach spaces.

Property: $\mathcal{C}$-injectivity

Let Y be a Banach space over field $\mathbb{K}$ and $Z \subseteq Y$ its closed subspace such that $Z \in \mathcal{C}$. If $\Phi : Z \to X$ is a bounded linear operator, then there exists $\bar{\Phi} : Y \to X$, a bounded linear extension of Φ.
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- universally separably injective $\to$ separable spaces
- universally κ-injective $\to$ spaces of density $< \kappa$
- universally Y-injective $\to$ spaces isomorphic to Y

Bounds

Theorem (refined)

If $2^\kappa > \mathfrak{c}$, then ℓ_∞ / c_0 is not universally $c_0(\kappa)$-injective, hence also not universally κ^+-injective.

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Define

$$\mathfrak{e} = \min\{\kappa : \ell_\infty/c_0 \text{ is not universally } \kappa^+\text{-injective}\}.$$

It is the smallest κ such that there is no extension of some operator with domain of density κ to some superspace of the domain.

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$$\omega_1 \leq \mathfrak{e} \leq \min\{\kappa : 2^\kappa > \mathfrak{c}\} \leq \mathfrak{c}$$

Another bound

$$\ell_\infty^c(\kappa) = \{x \in \ell_\infty(\kappa) : \mathrm{supp}(x) \text{ is at most countable}\}$$

Theorem (Koszmider, R. 2025)

If $\mathcal{A} := \{A_\xi : \xi < \kappa\}$ is a maximal almost disjoint family then there is no bounded operator $S : \ell_\infty^c(\kappa) \to \ell_\infty / c_0$ such that $S(1_{\{\xi\}}) = [1_{A_\xi}]$.

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$$\omega_1 \leq \mathfrak{e} \leq \mathfrak{a}, \min\{\kappa : 2^\kappa > \mathfrak{c}\} \leq \mathfrak{c}$$

Question: $\mathfrak{e} = \omega_1$?

κ^+-injectivity vs $c_0(\kappa)$-injectivity

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Are this bounds "optimal"? Maybe always $\mathfrak{e}_0 = \omega_1$?

Extensions of operators from $c_0(\kappa)$

Theorem (Koszmider, R. 2025)

Under MA(σ-centered) every isomorphism between two subspaces of ℓ_∞/c_0 isomorphic to $c_0(\kappa)$, $\kappa < \mathfrak{c}$, can be extended to an automorphism of ℓ_∞/c_0.

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The following is an application of a result of Moreno and Plichko:

Corollary

Under MA(σ-centered) every operator $T : Y \to \ell_\infty/c_0$ where $Y \subseteq \ell_\infty/c_0$ is a subspace isomorphic to $c_0(\kappa)$, $\kappa < \mathfrak{c}$, can be extended to an operator on the whole space.

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- for $\delta > 0$ small enough, $Id|Y + \delta T : Y \to \ell_\infty/c_0$ is bounded from below, hence an isomorphism onto its image

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- $\bar{T} := \delta^{-1}(U - Id)$ is a desired extension of T

$c_0(\kappa)$-injectivity of ℓ_∞/c_0

Theorem (Baumgartner, Frankiewicz, Zbierski)

There is a model of ZFC with arbitrarily large $\mathfrak{c}$ in which MA(σ-linked) holds and for every $\kappa < \mathfrak{c}$ space $\ell_\infty(\kappa)$ embeds into ℓ_∞/c_0.

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It is consistent with arbitrarily large $\mathfrak{c}$ that ℓ_∞/c_0 is universally $c_0(\kappa)$-injective for every $\kappa < \mathfrak{c}$.

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Conclusion

Consistently, $\mathfrak{e}_0 \neq \omega_1$.

Comments and questions

To resume:

- $\omega_1 \leq \mathfrak{e} \leq \mathfrak{a}, \min\{\kappa : 2^\kappa > \mathfrak{c}\}$
- $\omega_1 \leq \mathfrak{e}_0 \leq \mathfrak{a}, \min\{\kappa : 2^\kappa > \mathfrak{c}\}$ and consistently, $\mathfrak{e}_0 > \omega_1$

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1. Is ℓ_∞ / c_0 consistently (universally) $\aleph_2$-injective?

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Questions:

1. Is ℓ_∞ / c_0 consistently (universally) $\aleph_2$-injective?
2. Does $c_0(\kappa)$-injectivity imply κ^+-injectivity?

Theorem (Avilés, Cabello Sánchez, Castillo, González, Moreno)

A Banach space X is universally separably injective iff every separable subspace of X is contained in an isomorphic copy of ℓ_∞.

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1. Is ℓ_∞/c_0 consistently (universally) $\aleph_2$-injective?
2. Does $c_0(\kappa)$-injectivity imply κ^+-injectivity?

Theorem (Avilés, Cabello Sánchez, Castillo, González, Moreno)

A Banach space X is universally separably injective iff every separable subspace of X is contained in an isomorphic copy of ℓ_∞.

3. Suppose that space X is universally κ-injective. Does that imply that every subspace of X of density $< \kappa$ is contained in a copy of some injective space?

Comments and questions

To resume:

- $\omega_1 \leq \mathfrak{e} \leq \mathfrak{a}, \min\{\kappa : 2^\kappa > \mathfrak{c}\}$
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Thank you for your attention!

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