

Bounds on injectivity of ℓ_∞/c_0

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Hahn-Banach theorem

Let Y be a Banach space over field $\mathbb{K}$ and $Z \subseteq Y$ its closed subspace. If $\Phi : Z \rightarrow \mathbb{K}$ is a bounded linear functional, then there exists $\bar{\Phi} : Y \rightarrow \mathbb{K}$, a bounded linear extension of Φ (*such that* $\|\bar{\Phi}\| = \|\Phi\|$).

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- $\ell_\infty(K)$ is injective
- spaces $C(K)$ for K extremally disconnected are injective
- spaces isomorphic to an injective space are also injective
- anything else? (we don't know)

- $f \sim g \iff f - g \in c_0$ for $f, g \in \ell_\infty$
- $[f] = \{g \in \ell_\infty : f \sim g\}$
- $\ell_\infty/c_0 = \{[f] : f \in \ell_\infty\}$ with the quotient norm
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Observation

If $\mathcal{A} \subseteq \wp(\mathbb{N})$ is an almost disjoint family, then $Y_{\mathcal{A}} := \overline{\text{span}}\{[1_A] : A \in \mathcal{A}\}$ is an isometric copy of $c_0(|\mathcal{A}|)$ in ℓ_∞/c_0 .

Theorem (Rosenthal)

Let X be a Banach space, Γ an infinite set and $T : \ell_\infty(\Gamma) \rightarrow X$ an operator such that $T|_{c_0(\Gamma)}$ is an isomorphism. Then there is a set $\Gamma' \subseteq \Gamma$ with $|\Gamma'| = |\Gamma|$ such that $T|_{\ell_\infty(\Gamma')}$ is an isomorphism.

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ℓ_∞/c_0 is not injective.

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Proof:

- there is an almost disjoint family $\mathcal{A} \subseteq \wp(\mathbb{N})$ of cardinality $\mathfrak{c}$
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- if S extends to an operator $T : \ell_\infty(\mathfrak{c}) \rightarrow \ell_\infty/c_0$, then there is $\Gamma' \subseteq \mathfrak{c}$ with $|\Gamma'| = \mathfrak{c}$ such that $T|_{\ell_\infty(\Gamma')}$ is an isomorphism

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- if S extends to an operator $T : \ell_\infty(\mathfrak{c}) \rightarrow \ell_\infty/c_0$, then there is $\Gamma' \subseteq \mathfrak{c}$ with $|\Gamma'| = \mathfrak{c}$ such that $T|_{\ell_\infty(\Gamma')}$ is an isomorphism
- but $|\ell_\infty(\Gamma')| = 2^{\mathfrak{c}} > \mathfrak{c} = |\ell_\infty/c_0|$, a contradiction.

Separable injectivity of ℓ_∞/c_0

Theorem (Avilés, Cabello Sánchez, Castillo, González, Moreno)

Every separable subspace of ℓ_∞/c_0 is contained in a subspace isometrically isomorphic to ℓ_∞ .

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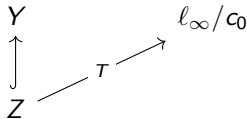
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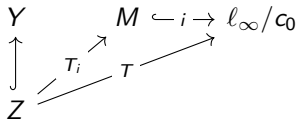
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There is a subspace $M \subseteq \ell_\infty/c_0$ isometrically isomorphic to ℓ_∞ which contains $T[Z]$.

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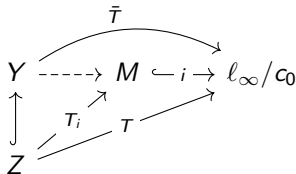
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There is a subspace $M \subseteq \ell_\infty/c_0$ isometrically isomorphic to ℓ_∞ which contains $T[Z]$. M is injective, so T can be extended.

Let $\mathcal{C}$ be a class of Banach spaces.

Property: $\mathcal{C}$ -injectivity

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Such X is said to be **universally $\mathcal{C}$ -injective**.

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- universally separably injective $\rightarrow$ separable spaces
- universally κ -injective $\rightarrow$ spaces of density $< \kappa$
- universally Y -injective $\rightarrow$ spaces isomorphic to Y

Theorem (refined)

If $2^\kappa > \mathfrak{c}$, then ℓ_∞/c_0 is not universally $c_0(\kappa)$ -injective, hence also not universally κ^+ -injective.

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Define

$$\mathfrak{e} = \min\{\kappa : \ell_\infty/c_0 \text{ is not universally } \kappa^+\text{-injective}\}.$$

It is the smallest κ such that there is no extension of some operator with domain of density κ to some superspace of the domain.

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$$\omega_1 \leq \mathfrak{e} \leq \min\{\kappa : 2^\kappa > \mathfrak{c}\} \leq \mathfrak{c}$$

$$\ell_{\infty}^c(\kappa) = \{x \in \ell_{\infty}(\kappa) : \text{supp}(x) \text{ is at most countable}\}$$

Theorem (Koszmider, R. 2025)

If $\mathcal{A} := \{A_{\xi} : \xi < \kappa\}$ is a maximal almost disjoint family then there is no bounded operator $S : \ell_{\infty}^c(\kappa) \rightarrow \ell_{\infty}/c_0$ such that $S(1_{\{\xi\}}) = [1_{A_{\xi}}]$.

Another bound

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Corollary

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Question: $\mathfrak{e} = \omega_1$?

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Are these bounds “optimal”? Maybe always $\mathfrak{e}_0 = \omega_1$?

Extensions of operators from $c_0(\kappa)$

Theorem (Koszmider, R. 2025)

Under $\text{MA}(\sigma\text{-centered})$ every isomorphism between two subspaces of ℓ_∞/c_0 isomorphic to $c_0(\kappa)$, $\kappa < \mathfrak{c}$, can be extended to an automorphism of ℓ_∞/c_0 .

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The following is an application of a result of Moreno and Plichko:

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Under $\text{MA}(\sigma\text{-centered})$ every operator $T : Y \rightarrow \ell_\infty/c_0$ where $Y \subseteq \ell_\infty/c_0$ is a subspace isomorphic to $c_0(\kappa)$, $\kappa < \mathfrak{c}$, can be extended to an operator on the whole space.

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- $\bar{T} := \delta^{-1}(U - Id)$ is a desired extension of T

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There is a model of ZFC with arbitrarily large $\mathfrak{c}$ in which $\text{MA}(\sigma\text{-linked})$ holds and for every $\kappa < \mathfrak{c}$ space $\ell_\infty(\kappa)$ embeds into ℓ_∞/c_0 .

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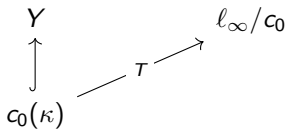
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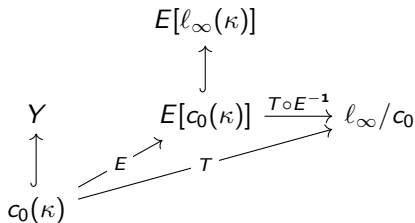
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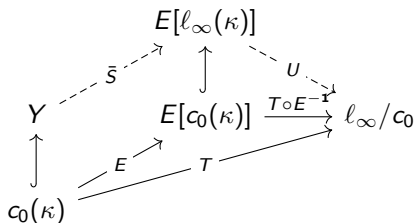
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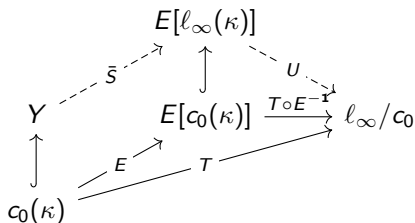
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Proof:



Conclusion

Consistently, $\mathfrak{e}_0 \neq \omega_1$.

To resume:

- $\omega_1 \leq \mathfrak{e} \leq \mathfrak{a}, \min\{\kappa : 2^\kappa > \mathfrak{c}\}$
- $\omega_1 \leq \mathfrak{e}_0 \leq \mathfrak{a}, \min\{\kappa : 2^\kappa > \mathfrak{c}\}$ and consistently, $\mathfrak{e}_0 > \omega_1$

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A Banach space X is universally separably injective iff every separable subspace of X is contained in an isomorphic copy of ℓ_∞ .

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- 3 Suppose that space X is universally κ -injective. Does that imply that every subspace of X of density $< \kappa$ is contained in a copy of some injective space?

To resume:

- $\omega_1 \leq \mathfrak{c} \leq \mathfrak{a}, \min\{\kappa : 2^\kappa > \mathfrak{c}\}$
- $\omega_1 \leq \mathfrak{e}_0 \leq \mathfrak{a}, \min\{\kappa : 2^\kappa > \mathfrak{c}\}$ and consistently, $\mathfrak{e}_0 > \omega_1$

Questions:

- 1 Is ℓ_∞/c_0 consistently (universally) $\aleph_2$ -injective?
- 2 Does $c_0(\kappa)$ -injectivity imply κ^+ -injectivity?

Theorem (Avilés, Cabello Sánchez, Castillo, González, Moreno)

A Banach space X is universally separably injective iff every separable subspace of X is contained in an isomorphic copy of ℓ_∞ .

- 3 Suppose that space X is universally κ -injective. Does that imply that every subspace of X of density $< \kappa$ is contained in a copy of some injective space?

Thank you for your attention!

- ① A. Avilés, F. Cabello Sánchez, J. Castillo, M. González, Y. Moreno, *On separably injective Banach spaces*
- ② J. Baumgartner, R. Frankiewicz, P. Zbierski, *Embedding of Boolean algebras in $\wp(\omega)/\text{fin}$.*
- ③ P. Koszmider, M. Rojek, *Almost disjoint families and some automorphic and injective properties of ℓ_∞/c_0*
- ④ Y. Moreno, A. Plichko, *On automorphic Banach spaces*
- ⑤ H. Rosenthal, *On relatively disjoint families of measures, with some applications to Banach space theory.*