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The packing problem in infinite dimensions

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j./w. C.A. De Bernardi, Ş. Sezgek, and J. Somaglia
Packings in classical Banach spaces, arXiv:2602.12934

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Packing problems

Kepler problem (1661)

What is the optimal way to stack cannonballs on a ship?

Gregory–Newton problem (1694)

Can a sphere in $\mathbb{R}^3$ touch 13 spheres of the same size?

- **Gregory**: Yes.
- **Newton**: No, at most 12.
- δ_n: the density of the optimal packing of spheres in $\mathbb{R}^n$.
- δ_n^*: with the extra requirement that the centers are a lattice (namely, a subgroup).

A few values

n	δ_n^*	Author	δ_n	Author
2	$\frac{\pi}{\sqrt{12}}$	Lagrange, 1773	$\frac{\pi}{\sqrt{12}}$	Thue, 1892
3	$\frac{\pi}{\sqrt{18}}$	Gauss, 1831	$\frac{\pi}{\sqrt{18}}$	Hales, 2005 *Annals*
4	$\frac{\pi^2}{16}$	Korkin, Zolotarev, 1872	???	
5	$\frac{\pi^2}{15\sqrt{2}}$	Korkin, Zolotarev, 1877	???	
6	$\frac{\pi^3}{48\sqrt{3}}$	Blichfeldt, 1925	???	
7	$\frac{\pi^3}{105}$	Blichfeldt, 1926	???	
8	$\frac{\pi^4}{384}$	Blichfeldt, 1934	$\frac{\pi^4}{384}$	Viazovska, 2017 *Annals*
24	$\frac{\pi^{12}}{12!}$	Cohn, Kumar, 2009 *Annals*	$\frac{\pi^{12}}{12!}$	Cohn et.al. 2017 *Annals*

- **Algorithm:** for infinitely many *Annals* papers.

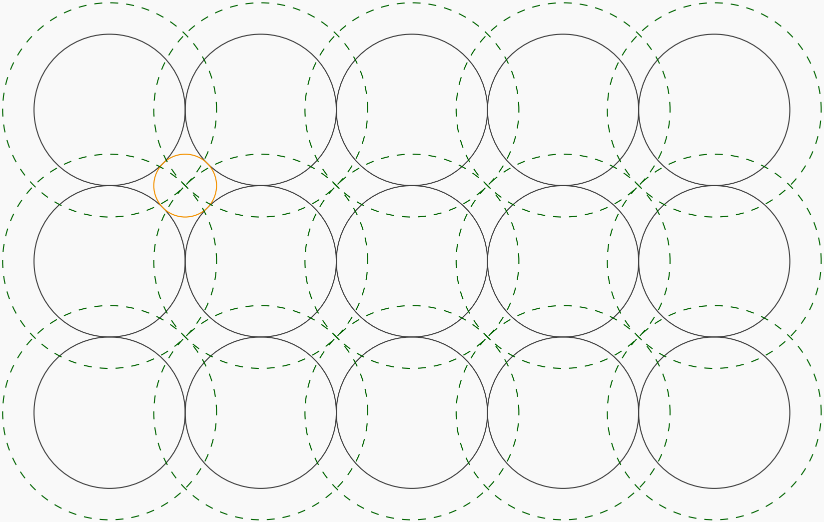
Beyond exact values

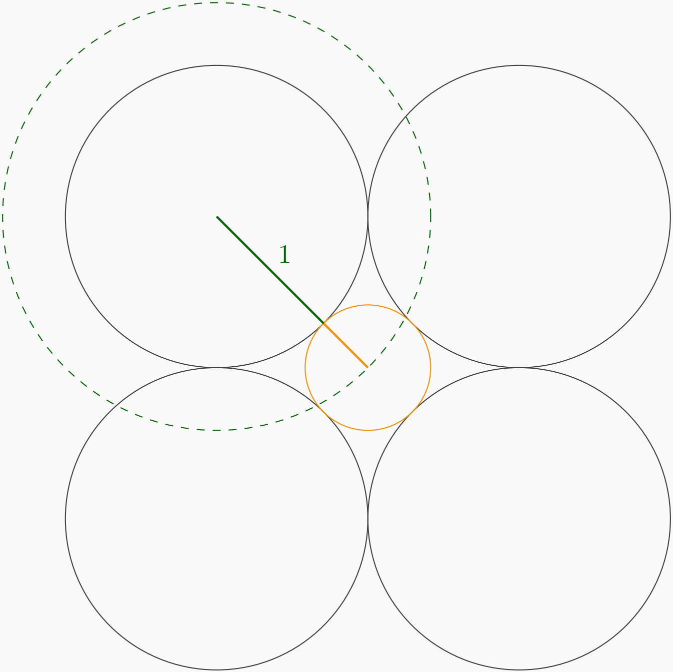
- **Klartag (2026).** $\delta_n^* \geqslant cn^2 2^{-n}$.
  - **Minkowski-Hlawka.** $\delta_n^* \geqslant c\zeta(n)2^{-n}$.
- **Kabatjanskiĭ, Levenšteĭn (1978).** $\delta_n \lesssim 0.66^n$.
- **Rogers (1964).** Conjecture that $\delta_n \neq \delta_n^*$ for some n.
- For a convex set C in $\mathbb{R}^n$, $\delta(C)$ is the optimal density of a packing by translates of C; and similarly for $\delta^*(C)$.
- **Fáry–Courant–Rogers.** For a convex set C in the plane,

  $$\delta(C) = \delta^*(C) \geqslant \frac{2}{3}$$

  with equality if and only if C is a triangle.
- **The definition can't be extended to infinite dimensions.**
- **Rogers (1950).** An alternative, purely metric, notion of optimality.

The simultaneous covering and packing constant

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The simultaneous covering and packing constant

- Compute the radius of the largest non-overlapping ball.
- How much do we have to inflate the balls to cover the space?
- First is second -1.
- The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ measures this (the second quantity).
- **Folklore.** $\gamma(\mathbb{R}^2) = \gamma^*(\mathbb{R}^2) = \frac{2}{\sqrt{3}}$.

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- **Folklore.** $\gamma(\mathbb{R}^2) = \gamma^*(\mathbb{R}^2) = \frac{2}{\sqrt{3}}$.
- **If $\mathcal{X}$ is finite dimensional:**
  - **Folklore.** $\gamma(\mathcal{X}) < 2$.
  - **Venkov–McMullen.** $\gamma(\mathcal{X}) = 1$ if and only if $\gamma^*(\mathcal{X}) = 1$.
  - **Mahler (1946).** $\gamma(\mathcal{X}) = 1$ if and only if $\mathcal{X}$ has a (lattice) tiling by balls of radius 1.
- **What about infinite-dimensional spaces?**

Two formulas

- The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ is

$$\gamma(\mathcal{X}) := \inf\{r > 0 \colon \exists\, \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense}\}.$$

- The **lattice simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ is

$$\gamma^*(\mathcal{X}) := \inf\{r > 0 \colon \exists\, \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense } \textbf{subgroup}\}.$$

Two formulas

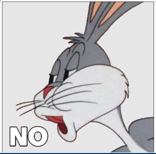
$$\gamma(\mathcal{X}) \coloneqq \inf\{r > 0 \colon \exists\, \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense}\}.$$

$$\gamma^*(\mathcal{X}) \coloneqq \inf\{r > 0 \colon \exists\, \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense } \textbf{subgroup}\}.$$

- If $\mathcal{X}$ has a packing that also covers, $\gamma(\mathcal{X}) = 1$.
- Taking any maximal packing $\rightsquigarrow \gamma(\mathcal{X}) \leqslant 2$.
- **De Bernardi, R., Somaglia (2026+).** $\gamma^*(\mathcal{X}) \leqslant 2$.
- **Casini, Papini, Zanco (1986).** $\gamma(\mathcal{X}) \geqslant \frac{2}{K(\mathcal{X})}$.
- To sum it up

$$\textbf{(good)} \quad 1 \leqslant \frac{2}{K(\mathcal{X})} \leqslant \gamma(\mathcal{X}) \leqslant \gamma^*(\mathcal{X}) \leqslant 2 \quad \textbf{(bad)}$$

- **Is it true that $\gamma(\mathcal{X}) < 2$ for all spaces?**
  - **Are we smarter than Zorn?**



A problem of Swanepoel

- **Our original motivation:**
- **Swanepoel (2009).** $\gamma(\ell_p) = \frac{2}{2^{1/p}}$ which equals $\frac{2}{K(\ell_p)}$.
- **Swanepoel (2009).** Is it true that for all Banach spaces

$$\gamma(\mathcal{X}) = \frac{2}{K(\mathcal{X})}?$$

- **If the unit ball of $\mathcal{X}$ admits a LUR point, then $\gamma(\mathcal{X}) > 1$.**
- Consider $\ell_1 \oplus_2 \mathbb{R}$. Its unit ball has a LUR point, but $K(\ell_1 \oplus_2 \mathbb{R}) = 2$.
- Every Banach space $\mathcal{X}$ is isomorphic to a Banach space $\mathcal{Y}$ with $K(\mathcal{Y}) = 2$ and $\gamma(\mathcal{Y}) > 1$.
  - **So, there are reflexive (even isomorphic to ℓ_2) counterexamples to Swanepoel's question.**

Some known values

- For $1 \leqslant p < \infty$ and every infinite κ

$$\gamma(\ell_p(\kappa)) = \gamma^*(\ell_p(\kappa)) = \frac{2}{2^{1/p}}.$$

- For $1 \leqslant p \leqslant 2$ and every measure μ (with $L_p(\mu)$ inf-dim)

$$\gamma(L_p(\mu)) = \gamma^*(L_p(\mu)) = \frac{2}{2^{1/p}}.$$

- If $\mathcal{X}$ is **separable** and octahedral, or $\mathcal{X} = \mathcal{C}(\mathcal{K})$ with $\mathcal{K}$ zero-dimensional

$$\gamma(\mathcal{X}) = \gamma^*(\mathcal{X}) = 1.$$

- This applies, *e.g.*, to: $L_1(\mu)$, $\mathcal{C}([0,1])$, $\mathcal{C}(2^\omega)$, $\mathcal{C}(\mathcal{K})$ for $\mathcal{K}$ countable (or scattered).
- Some Lipschitz-free spaces, spaces of Lipschitz functions, tensor products, ...

Lower bounds

- A **modulus** is a function $\phi\colon (0,\infty) \to (0,\infty)$ such that $\phi(0^+) = 0$ and $t \mapsto \frac{\phi(t)}{t}$ is increasing.
- A normed space $\mathcal{X}$ is **ϕ-octahedral** if for every $\varepsilon > 0$ and every finite-dimensional subspace $\mathcal{Z}$ of $\mathcal{X}$ there is $x \in S_{\mathcal{X}}$ with

$$\|z + \lambda x\| \geqslant (1 - \varepsilon)\big(1 + \phi(|\lambda|)\big) \qquad z \in S_{\mathcal{Z}},\ \lambda \neq 0.$$

- Octahedral spaces are ϕ-octahedral; **so are uniformly convex ones**.
- **Take $p \in [1,\infty)$, $p_k \to \infty$. Then**

$$\mathcal{X}_p = \left(\bigoplus_{k=1}^{\infty} \ell_{p_k}(\omega_k) \right)_{\ell_p}$$

  satisfies $\gamma(\mathcal{X}_p) = 2$.
- **There are reflexive (resp. octahedral) spaces with $\gamma(\mathcal{X}) = 2$.**

Open problems

- Is there a **separable** Banach space $\mathcal{X}$ with $\gamma(\mathcal{X}) = 2$?
  - Or a **super-reflexive** one?
- Is there a normed space $\mathcal{X}$ with $\gamma(\mathcal{X}) \neq \gamma^*(\mathcal{X})$?
- What are the exact values of $\gamma(\ell_1 \oplus_2 \mathbb{R})$ and $\gamma^*(\ell_1 \oplus_2 \mathbb{R})$?
  - They are > 1 (LUR point).
- What are the exact values of $\gamma(\ell_1 \oplus_2 \ell_1)$ and $\gamma^*(\ell_1 \oplus_2 \ell_1)$?
- **And many more...**

Thank you for your attention!