Bourgain-Rosenthal-Schechtman spaces in rearrangement-invariant and Hardy spaces

Thomas Speckhofer

York University

Joint work with Konstantinos Konstantos and Pavlos Motakis

Summer School: Topics in Banach Space Theory

July 9, 2026

Introduction

- In 1981, Bourgain, Rosenthal, and Schechtman constructed uncountably many pairwise non-isomorphic *complemented* subspaces of L^p, $1 < p < \infty$, $p \neq 2$.
- Applications: e.g., number of closed operator ideals on L^p
- Ghawadrah 2016: extended results to reflexive Orlicz function spaces
- Konstantos-Motakis: factorization results in BRS spaces
- Goal: Extend original BRS result to more general rearrangement-invariant spaces and Hardy spaces.

Introduction

- In 1981, Bourgain, Rosenthal, and Schechtman constructed uncountably many pairwise non-isomorphic *complemented* subspaces of L^p, $1 < p < \infty$, $p \neq 2$.
- Applications: e.g., number of closed operator ideals on L^p
- Ghawadrah 2016: extended results to reflexive Orlicz function spaces
- Konstantos-Motakis: factorization results in BRS spaces
- Goal: Extend original BRS result to more general rearrangement-invariant spaces and Hardy spaces.

Introduction

- In 1981, Bourgain, Rosenthal, and Schechtman constructed uncountably many pairwise non-isomorphic *complemented* subspaces of L^p, $1 < p < \infty$, $p \neq 2$.
- Applications: e.g., number of closed operator ideals on L^p
- Ghawadrah 2016: extended results to reflexive Orlicz function spaces
- Konstantos-Motakis: factorization results in BRS spaces
- Goal: Extend original BRS result to more general rearrangement-invariant spaces and Hardy spaces.

Introduction

- In 1981, Bourgain, Rosenthal, and Schechtman constructed uncountably many pairwise non-isomorphic *complemented* subspaces of L^p, $1 < p < \infty$, $p \neq 2$.
- Applications: e.g., number of closed operator ideals on L^p
- Ghawadrah 2016: extended results to reflexive Orlicz function spaces
- Konstantos-Motakis: factorization results in BRS spaces
- Goal: Extend original BRS result to more general rearrangement-invariant spaces and Hardy spaces.

Introduction

- In 1981, Bourgain, Rosenthal, and Schechtman constructed uncountably many pairwise non-isomorphic *complemented* subspaces of L^p, $1 < p < \infty$, $p \neq 2$.
- Applications: e.g., number of closed operator ideals on L^p
- Ghawadrah 2016: extended results to reflexive Orlicz function spaces
- Konstantos-Motakis: factorization results in BRS spaces
- Goal: Extend original BRS result to more general rearrangement-invariant spaces and Hardy spaces.

Definitions

- Dyadic intervals: $\mathcal{D} = \{[0,1), [0,\frac{1}{2}), [\frac{1}{2},1), [0,\frac{1}{4}), [\frac{1}{4},\frac{1}{2}), \dots\}$
- The Haar function h_I is 1 on the left and -1 on the right half of I.
- Together with $\mathbb{1}_{[0,1)}$, the Haar system $(h_I)_{I\in\mathcal{D}}$ is a monotone Schauder basis of L^p, $1 \le p < \infty$.
- Rademacher functions: $r_n = \sum_{|K|=2^{-n}} h_K$ for $n \in \mathbb{N}_0$.

Definitions

- Dyadic intervals: $\mathcal{D} = \{[0,1), [0,\frac{1}{2}), [\frac{1}{2},1), [0,\frac{1}{4}), [\frac{1}{4},\frac{1}{2}), \dots\}$
- The Haar function h_I is 1 on the left and -1 on the right half of I.
- Together with $\mathbb{1}_{[0,1)}$, the Haar system $(h_I)_{I\in\mathcal{D}}$ is a monotone Schauder basis of L^p, $1 \le p < \infty$.
- Rademacher functions: $r_n = \sum_{|K|=2^{-n}} h_K$ for $n \in \mathbb{N}_0$.

Definitions

- Dyadic intervals: $\mathcal{D} = \{[0,1), [0,\frac{1}{2}), [\frac{1}{2},1), [0,\frac{1}{4}), [\frac{1}{4},\frac{1}{2}), \dots\}$
- The Haar function h_I is 1 on the left and -1 on the right half of I.
- Together with $\mathbb{1}_{[0,1)}$, the Haar system $(h_I)_{I\in\mathcal{D}}$ is a monotone Schauder basis of L^p, $1 \le p < \infty$.
- Rademacher functions: $r_n = \sum_{|K|=2^{-n}} h_K$ for $n \in \mathbb{N}_0$.

Definitions

- Dyadic intervals: $\mathcal{D} = \{[0,1), [0,\frac{1}{2}), [\frac{1}{2},1), [0,\frac{1}{4}), [\frac{1}{4},\frac{1}{2}), \dots\}$
- The Haar function h_I is 1 on the left and -1 on the right half of I.
- Together with $\mathbb{1}_{[0,1)}$, the Haar system $(h_I)_{I\in\mathcal{D}}$ is a monotone Schauder basis of L^p, $1 \leq p < \infty$.
- Rademacher functions: $r_n = \sum_{|K|=2^{-n}} h_K$ for $n \in \mathbb{N}_0$.

Definitions

- A **Haar system Hardy space** Y is the completion of $\operatorname{span}\{h_I : I \in \mathcal{D}\}$ under a rearrangement-invariant norm $\|\cdot\|_X$ (i.e., $\operatorname{dist}(|f|) = \operatorname{dist}(|g|) \implies \|f\|_X = \|g\|_X$)

  or under the square function norm $\|\cdot\|_{H_X}$ given by

$$\Big\| \sum_{I \in \mathcal{D}} a_I h_I \Big\|_{H_X} = \Big\| \Big(\sum_{I \in \mathcal{D}} a_I^2 h_I^2 \Big)^{1/2} \Big\|_X .$$

- X ... Haar system space, H_X ... associated Hardy space
- Examples: L_0^p, $1 \leq p < \infty$, all separable rearrangement-invariant function spaces, dyadic Hardy space H^1.

Definitions

- A **Haar system Hardy space** Y is the completion of $\operatorname{span}\{h_I : I \in \mathcal{D}\}$ under a rearrangement-invariant norm $\|\cdot\|_X$ (i.e., $\operatorname{dist}(|f|) = \operatorname{dist}(|g|) \implies \|f\|_X = \|g\|_X$)

  or under the square function norm $\|\cdot\|_{H_X}$ given by

$$\Big\|\sum_{I\in\mathcal{D}} a_I h_I\Big\|_{H_X} = \Big\|\Big(\sum_{I\in\mathcal{D}} a_I^2 h_I^2\Big)^{1/2}\Big\|_X.$$

- X ... Haar system space, H_X ... associated Hardy space
- Examples: L_0^p, $1 \le p < \infty$, all separable rearrangement-invariant function spaces, dyadic Hardy space H^1.

Definitions

- A **Haar system Hardy space** Y is the completion of $\operatorname{span}\{h_I : I \in \mathcal{D}\}$ under a rearrangement-invariant norm $\|\cdot\|_X$ (i.e., $\operatorname{dist}(|f|) = \operatorname{dist}(|g|) \implies \|f\|_X = \|g\|_X$)

  or under the square function norm $\|\cdot\|_{H_X}$ given by

$$\Big\|\sum_{I\in\mathcal{D}} a_I h_I\Big\|_{H_X} = \Big\|\Big(\sum_{I\in\mathcal{D}} a_I^2 h_I^2\Big)^{1/2}\Big\|_X.$$

- X ... Haar system space, H_X ... associated Hardy space
- Examples: L_0^p, $1 \le p < \infty$, all separable rearrangement-invariant function spaces, dyadic Hardy space H^1.

Definitions

- A **Haar system Hardy space** Y is the completion of $\operatorname{span}\{h_I : I \in \mathcal{D}\}$ under a rearrangement-invariant norm $\|\cdot\|_X$ (i.e., $\operatorname{dist}(|f|) = \operatorname{dist}(|g|) \implies \|f\|_X = \|g\|_X$)

  or under the square function norm $\|\cdot\|_{H_X}$ given by

$$\Big\|\sum_{I\in\mathcal{D}} a_I h_I\Big\|_{H_X} = \Big\|\Big(\sum_{I\in\mathcal{D}} a_I^2 h_I^2\Big)^{1/2}\Big\|_X.$$

- X ... Haar system space, H_X ... associated Hardy space
- Examples: L_0^p, $1 \le p < \infty$, all separable rearrangement-invariant function spaces, dyadic Hardy space H^1.

The Bourgain-Rosenthal-Schechtman spaces

Now fix a Haar system Hardy space Y.

- Construct R_α^Y, $0 \leq \alpha < \omega_1$ by alternately taking **disjoint** and **independent** sums:
  - $R_0^Y = \{0\}$
  - $R_{\alpha+1}^Y = \langle \{h_{[0,1)}\} \rangle + (R_\alpha^Y \oplus R_\alpha^Y)_{\mathrm{disj}}$
  - $R_\alpha^Y = (\oplus_{\beta<\alpha} R_\beta^Y)_{Y,\mathrm{ind}}$ if α is limit ordinal
- Disjoint sum: $(Z \oplus W)_{\mathrm{disj}} = T_{[0,\frac{1}{2})}(Z) + T_{[\frac{1}{2},1)}(W)$, where $(T_{[a,b)}f)(t) = f(\frac{t-a}{b-a})$ on $[a,b)$
- Independent sum: $(\oplus_{n=1}^\infty Z_n)_{Y,\mathrm{ind}}$ = closed span of certain *independent* distributional copies of Z_n, $n \in \mathbb{N}$.

The Bourgain-Rosenthal-Schechtman spaces

Now fix a Haar system Hardy space Y.

- Construct R^Y_α, $0 \leq \alpha < \omega_1$ by alternately taking **disjoint** and **independent** sums:
  - $R^Y_0 = \{0\}$
  - $R^Y_{\alpha+1} = \langle\{h_{[0,1)}\}\rangle + (R^Y_\alpha \oplus R^Y_\alpha)_{\mathrm{disj}}$
  - $R^Y_\alpha = (\oplus_{\beta<\alpha} R^Y_\beta)_{Y,\mathrm{ind}}$ if α is limit ordinal
- Disjoint sum: $(Z \oplus W)_{\mathrm{disj}} = T_{[0,\frac{1}{2})}(Z) + T_{[\frac{1}{2},1)}(W)$, where $(T_{[a,b)}f)(t) = f(\frac{t-a}{b-a})$ on $[a,b)$
- Independent sum: $(\oplus_{n=1}^\infty Z_n)_{Y,\mathrm{ind}}$ = closed span of certain *independent* distributional copies of Z_n, $n \in \mathbb{N}$.

The Bourgain-Rosenthal-Schechtman spaces

Now fix a Haar system Hardy space Y.

- Construct R_α^Y, $0 \leq \alpha < \omega_1$ by alternately taking **disjoint** and **independent** sums:
  - $R_0^Y = \{0\}$
  - $R_{\alpha+1}^Y = \langle\{h_{[0,1)}\}\rangle + (R_\alpha^Y \oplus R_\alpha^Y)_{\mathrm{disj}}$
  - $R_\alpha^Y = (\oplus_{\beta<\alpha} R_\beta^Y)_{Y,\mathrm{ind}}$ if α is limit ordinal
- Disjoint sum: $(Z \oplus W)_{\mathrm{disj}} = T_{[0,\frac{1}{2})}(Z) + T_{[\frac{1}{2},1)}(W)$, where $(T_{[a,b)}f)(t) = f(\frac{t-a}{b-a})$ on $[a,b)$
- Independent sum: $(\oplus_{n=1}^\infty Z_n)_{Y,\mathrm{ind}}$ = closed span of certain *independent* distributional copies of Z_n, $n \in \mathbb{N}$.

The Bourgain-Rosenthal-Schechtman spaces

Now fix a Haar system Hardy space Y.

- Construct R_α^Y, $0 \leq \alpha < \omega_1$ by alternately taking **disjoint** and **independent** sums:
  - $R_0^Y = \{0\}$
  - $R_{\alpha+1}^Y = \langle \{h_{[0,1)}\} \rangle + (R_\alpha^Y \oplus R_\alpha^Y)_{\mathrm{disj}}$
  - $R_\alpha^Y = (\oplus_{\beta<\alpha} R_\beta^Y)_{Y,\mathrm{ind}}$ if α is limit ordinal
- Disjoint sum: $(Z \oplus W)_{\mathrm{disj}} = T_{[0,\frac{1}{2})}(Z) + T_{[\frac{1}{2},1)}(W)$, where $(T_{[a,b)}f)(t) = f(\frac{t-a}{b-a})$ on $[a,b)$
- Independent sum: $(\oplus_{n=1}^\infty Z_n)_{Y,\mathrm{ind}}$ = closed span of certain *independent* distributional copies of Z_n, $n \in \mathbb{N}$.

The Bourgain-Rosenthal-Schechtman spaces

Now fix a Haar system Hardy space Y.

- Construct R^Y_α, $0 \leq \alpha < \omega_1$ by alternately taking **disjoint** and **independent** sums:
  - $R^Y_0 = \{0\}$
  - $R^Y_{\alpha+1} = \langle\{h_{[0,1)}\}\rangle + (R^Y_\alpha \oplus R^Y_\alpha)_{\mathrm{disj}}$
  - $R^Y_\alpha = (\oplus_{\beta<\alpha} R^Y_\beta)_{Y,\mathrm{ind}}$ if α is limit ordinal
- Disjoint sum: $(Z \oplus W)_{\mathrm{disj}} = T_{[0,\frac{1}{2})}(Z) + T_{[\frac{1}{2},1)}(W)$, where $(T_{[a,b)}f)(t) = f(\frac{t-a}{b-a})$ on $[a,b)$
- Independent sum: $(\oplus_{n=1}^\infty Z_n)_{Y,\mathrm{ind}}$ = closed span of certain *independent* distributional copies of Z_n, $n \in \mathbb{N}$.

The Bourgain-Rosenthal-Schechtman spaces

Now fix a Haar system Hardy space Y.

- Construct R_α^Y, $0 \leq \alpha < \omega_1$ by alternately taking **disjoint** and **independent** sums:
  - $R_0^Y = \{0\}$
  - $R_{\alpha+1}^Y = \langle\{h_{[0,1)}\}\rangle + (R_\alpha^Y \oplus R_\alpha^Y)_{\mathrm{disj}}$
  - $R_\alpha^Y = (\oplus_{\beta<\alpha} R_\beta^Y)_{Y,\mathrm{ind}}$ if α is limit ordinal
- Disjoint sum: $(Z \oplus W)_{\mathrm{disj}} = T_{[0,\frac{1}{2})}(Z) + T_{[\frac{1}{2},1)}(W)$, where $(T_{[a,b)}f)(t) = f(\frac{t-a}{b-a})$ on $[a,b)$
- Independent sum: $(\oplus_{n=1}^\infty Z_n)_{Y,\mathrm{ind}}$ = closed span of certain *independent* distributional copies of Z_n, $n \in \mathbb{N}$.

The Bourgain-Rosenthal-Schechtman spaces

Now fix a Haar system Hardy space Y.

- Construct R_α^Y, $0 \leq \alpha < \omega_1$ by alternately taking **disjoint** and **independent** sums:
  - $R_0^Y = \{0\}$
  - $R_{\alpha+1}^Y = \langle\{h_{[0,1)}\}\rangle + (R_\alpha^Y \oplus R_\alpha^Y)_{\mathrm{disj}}$
  - $R_\alpha^Y = (\oplus_{\beta<\alpha} R_\beta^Y)_{Y,\mathrm{ind}}$ if α is limit ordinal
- Disjoint sum: $(Z \oplus W)_{\mathrm{disj}} = T_{[0,\frac{1}{2})}(Z) + T_{[\frac{1}{2},1)}(W)$, where $(T_{[a,b)}f)(t) = f(\frac{t-a}{b-a})$ on $[a,b)$
- Independent sum: $(\oplus_{n=1}^{\infty} Z_n)_{Y,\mathrm{ind}}$ = closed span of certain *independent* distributional copies of Z_n, $n \in \mathbb{N}$.

The Bourgain-Rosenthal-Schechtman spaces

Now fix a Haar system Hardy space Y.

- Construct R_α^Y, $0 \leq \alpha < \omega_1$ by alternately taking **disjoint** and **independent** sums:
  - $R_0^Y = \{0\}$
  - $R_{\alpha+1}^Y = \langle\{h_{[0,1)}\}\rangle + (R_\alpha^Y \oplus R_\alpha^Y)_{\mathrm{disj}}$
  - $R_\alpha^Y = (\oplus_{\beta<\alpha} R_\beta^Y)_{Y,\mathrm{ind}}$ if α is limit ordinal
- Disjoint sum: $(Z \oplus W)_{\mathrm{disj}} = T_{[0,\frac{1}{2})}(Z) + T_{[\frac{1}{2},1)}(W)$, where $(T_{[a,b)}f)(t) = f(\frac{t-a}{b-a})$ on $[a,b)$
- Independent sum: $(\oplus_{n=1}^\infty Z_n)_{Y,\mathrm{ind}} =$ closed span of certain *independent* distributional copies of Z_n, $n \in \mathbb{N}$.

Independent sums

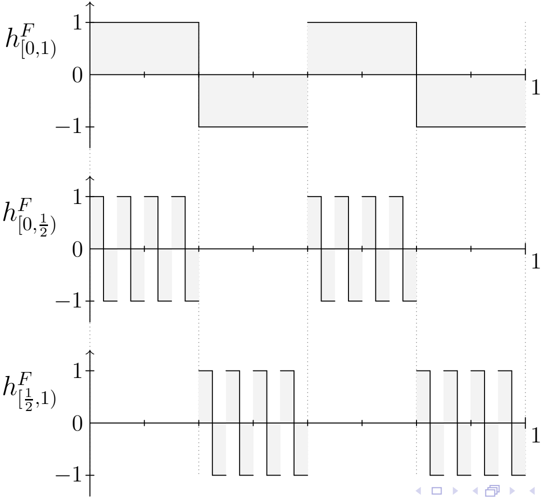
- For $F \subset \mathbb{N}$ infinite, construct a "Haar system with frequencies in F".
- Its closed span Y_F is isometrically isomorphic to Y:

Independent sums

- For $F \subset \mathbb{N}$ infinite, construct a "Haar system with frequencies in F".
- Its closed span Y_F is isometrically isomorphic to Y:

Independent sums

- For $F \subset \mathbb{N}$ infinite, construct a "Haar system with frequencies in F".
- Its closed span Y_F is isometrically isomorphic to Y:



Independent sums

- Now split $\mathbb{N}$ into infinitely many pairwise disjoint subsets $F_1, F_2, \ldots$.
- Map Z_n into Y_{F_n} via the above isometric embedding (call it J_n).
- Define $(\oplus_{n=1}^{\infty} Z_n)_{Y,\mathrm{ind}} = [J_n(Z_n)]_{n=1}^{\infty}$.

Independent sums

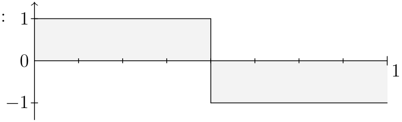
- Now split $\mathbb{N}$ into infinitely many pairwise disjoint subsets $F_1, F_2, \ldots$.
- Map Z_n into Y_{F_n} via the above isometric embedding (call it J_n).
- Define $(\oplus_{n=1}^{\infty} Z_n)_{Y,\mathrm{ind}} = [J_n(Z_n)]_{n=1}^{\infty}$.

Independent sums

- Now split $\mathbb{N}$ into infinitely many pairwise disjoint subsets $F_1, F_2, \ldots$.
- Map Z_n into Y_{F_n} via the above isometric embedding (call it J_n).
- Define $(\oplus_{n=1}^{\infty} Z_n)_{Y,\mathrm{ind}} = [J_n(Z_n)]_{n=1}^{\infty}$.

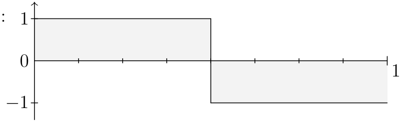
Bourgain-Rosenthal-Schechtman spaces

R_1^Y:

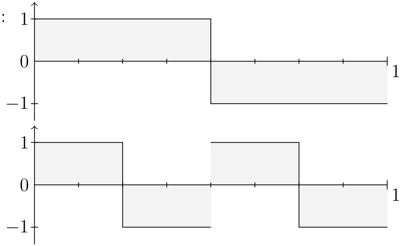


Bourgain-Rosenthal-Schechtman spaces

R_1^Y:

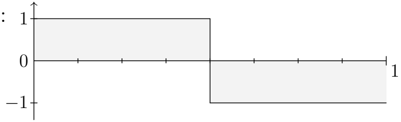


R_2^Y:

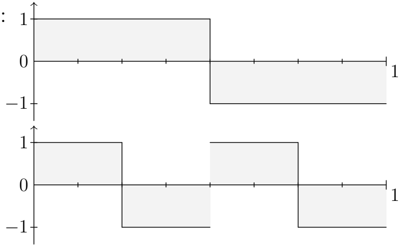


Bourgain-Rosenthal-Schechtman spaces

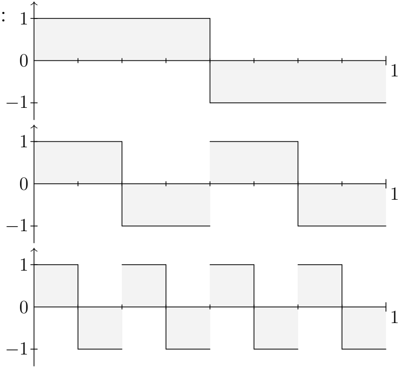
R_1^Y :



R_2^Y :



R_3^Y :



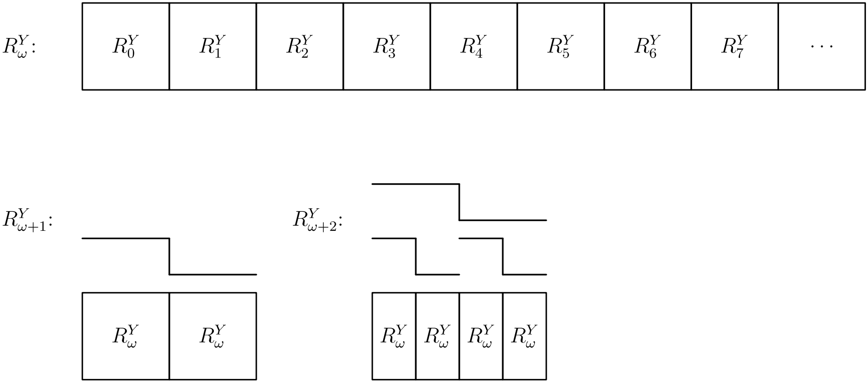
R^Y_ω:	R^Y_0	R^Y_1	R^Y_2	R^Y_3	R^Y_4	R^Y_5	R^Y_6	R^Y_7	$\cdots$

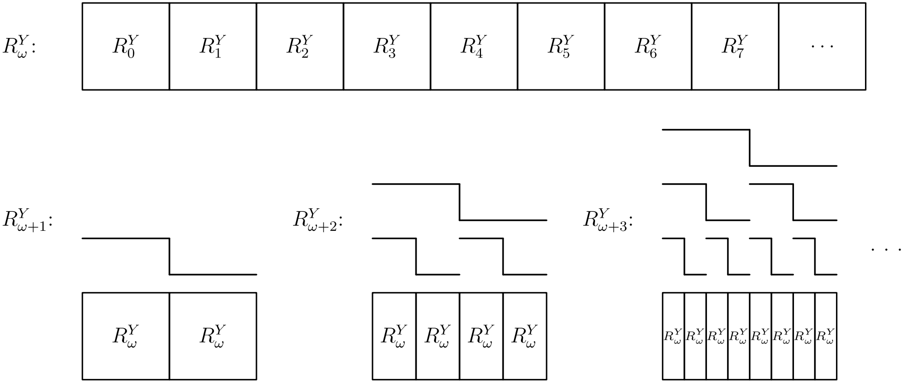
R^Y_ω:

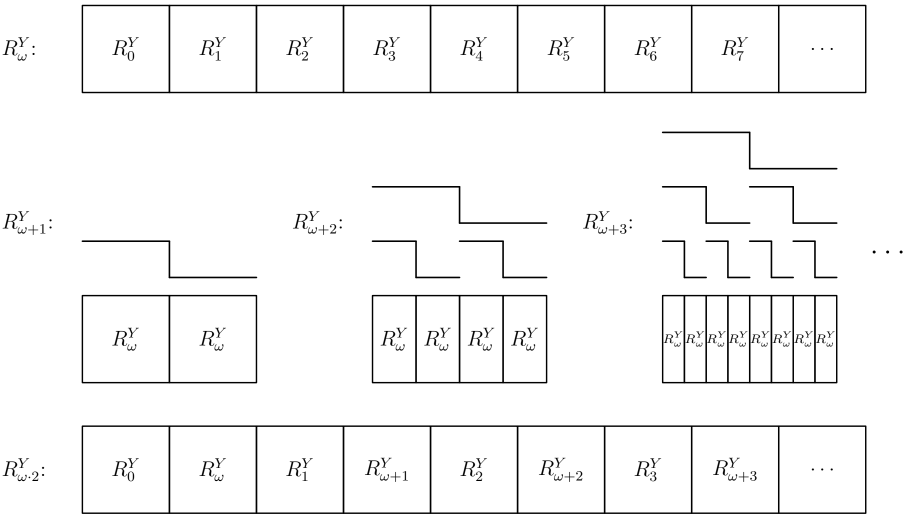
R^Y_0	R^Y_1	R^Y_2	R^Y_3	R^Y_4	R^Y_5	R^Y_6	R^Y_7	$\cdots$

$R^Y_{\omega+1}$:

R^Y_ω	R^Y_ω


R^Y_ω:
R^Y_0
R^Y_1
R^Y_2
R^Y_3
R^Y_4
R^Y_5
R^Y_6
R^Y_7
$\cdots$
$R^Y_{\omega+1}$:
R^Y_ω
R^Y_ω
$R^Y_{\omega+2}$:
R^Y_ω
R^Y_ω
R^Y_ω
R^Y_ω


R^Y_ω:
R^Y_0 R^Y_1 R^Y_2 R^Y_3 R^Y_4 R^Y_5 R^Y_6 R^Y_7 $\cdots$
$R^Y_{\omega+1}$:
R^Y_ω R^Y_ω
$R^Y_{\omega+2}$:
R^Y_ω R^Y_ω R^Y_ω R^Y_ω
$R^Y_{\omega+3}$:
R^Y_ω R^Y_ω R^Y_ω R^Y_ω R^Y_ω R^Y_ω R^Y_ω R^Y_ω
$\cdots$


R^Y_ω:
R^Y_0 R^Y_1 R^Y_2 R^Y_3 R^Y_4 R^Y_5 R^Y_6 R^Y_7 $\cdots$
$R^Y_{\omega+1}$:
R^Y_ω R^Y_ω
$R^Y_{\omega+2}$:
R^Y_ω R^Y_ω R^Y_ω R^Y_ω
$R^Y_{\omega+3}$:
R^Y_ω R^Y_ω R^Y_ω R^Y_ω R^Y_ω R^Y_ω R^Y_ω R^Y_ω
$\cdots$
$R^Y_{\omega\cdot 2}$:
R^Y_0 R^Y_ω R^Y_1 $R^Y_{\omega+1}$ R^Y_2 $R^Y_{\omega+2}$ R^Y_3 $R^Y_{\omega+3}$ $\cdots$

Main result

Theorem (Konstantos, Motakis, S. '26)

Let X be a Haar system space with the Fatou property and non-trivial Boyd indices, and $Y = \{f \in X : \mathbb{E}f = 0\}$. Then all R_α^Y are complemented in Y, and one of the following exclusive alternatives holds:

- *There exists a strictly increasing transfinite sequence of countable ordinals $(\alpha_\xi)_{\xi<\omega_1}$ such that the spaces $R_{\alpha_\xi}^Y$, $\xi < \omega_1$, are pairwise non-isomorphic.*
- *For every ordinal $\omega \le \alpha < \omega_1$, the space R_α^Y is isomorphic to Y.*

- Non-trivial Boyd indices = "not close to L^1 or L^∞".
- The second alternative is false in L^p, $1 < p < \infty$, except in L^2. Are there other examples where it is true?

Main result

Theorem (Konstantos, Motakis, S. '26)

Let X be a Haar system space with the Fatou property and non-trivial Boyd indices, and $Y = \{f \in X : \mathbb{E}f = 0\}$. Then all R_α^Y are complemented in Y, and one of the following exclusive alternatives holds:

- *There exists a strictly increasing transfinite sequence of countable ordinals $(\alpha_\xi)_{\xi<\omega_1}$ such that the spaces $R_{\alpha_\xi}^Y$, $\xi < \omega_1$, are pairwise non-isomorphic.*
- *For every ordinal $\omega \leq \alpha < \omega_1$, the space R_α^Y is isomorphic to Y.*

- Non-trivial Boyd indices = "not close to L^1 or L^∞".
- The second alternative is false in L^p, $1 < p < \infty$, except in L^2. Are there other examples where it is true?

Main result

Theorem (Konstantos, Motakis, S. '26)

Let X be a Haar system space with the Fatou property and non-trivial Boyd indices, and $Y = \{f \in X : \mathbb{E}f = 0\}$. Then all R_α^Y are complemented in Y, and one of the following exclusive alternatives holds:

- *There exists a strictly increasing transfinite sequence of countable ordinals $(\alpha_\xi)_{\xi<\omega_1}$ such that the spaces $R_{\alpha_\xi}^Y$, $\xi < \omega_1$, are pairwise non-isomorphic.*
- *For every ordinal $\omega \leq \alpha < \omega_1$, the space R_α^Y is isomorphic to Y.*

- Non-trivial Boyd indices = "not close to L^1 or L^∞".
- The second alternative is false in L^p, $1 < p < \infty$, except in L^2. Are there other examples where it is true?

Main result

Theorem (Konstantos, Motakis, S. '26)

Let X be a Haar system space with the Fatou property and non-trivial Boyd indices, and $Y = \{f \in X : \mathbb{E}f = 0\}$. Then all R_α^Y are complemented in Y, and one of the following exclusive alternatives holds:

- *There exists a strictly increasing transfinite sequence of countable ordinals $(\alpha_\xi)_{\xi<\omega_1}$ such that the spaces $R_{\alpha_\xi}^Y$, $\xi < \omega_1$, are pairwise non-isomorphic.*
- *For every ordinal $\omega \leq \alpha < \omega_1$, the space R_α^Y is isomorphic to Y.*

- Non-trivial Boyd indices $=$ "not close to L^1 or L^∞".
- The second alternative is false in L^p, $1 < p < \infty$, except in L^2. Are there other examples where it is true?

Main result

Theorem (Konstantos, Motakis, S. '26)

Let X be a Haar system space with the Fatou property and non-trivial Boyd indices, and $Y = \{f \in X : \mathbb{E}f = 0\}$. Then all R_α^Y are complemented in Y, and one of the following exclusive alternatives holds:

- *There exists a strictly increasing transfinite sequence of countable ordinals $(\alpha_\xi)_{\xi<\omega_1}$ such that the spaces $R_{\alpha_\xi}^Y$, $\xi < \omega_1$, are pairwise non-isomorphic.*
- *For every ordinal $\omega \leq \alpha < \omega_1$, the space R_α^Y is isomorphic to Y.*

- Non-trivial Boyd indices = "not close to L^1 or L^∞".
- The second alternative is false in L^p, $1 < p < \infty$, except in L^2. Are there other examples where it is true?

Proof

- First, suppose that for some α, R_α^Y contains a complemented subspace isomorphic to Y.
- The *smallest* such α is a limit ordinal, so $Y \overset{c}{\hookrightarrow} (\bigoplus_{\beta<\alpha} R_\beta^Y)_{Y,\mathrm{ind}}$, but $Y \overset{c}{\not\hookrightarrow} R_\beta^Y$ for all $\beta < \alpha$.
- Ghawadrah 2016 $\implies$ there is an **independent** sequence in R_α^Y equivalent to the Haar system with closed span complemented in Y.
- Then such a system exists in any R_β^Y, $\omega \leq \beta < \omega_1$.
- Hence, $Y \overset{c}{\hookrightarrow} R_\beta^Y$, $R_\beta^Y \overset{c}{\hookrightarrow} Y$ (by Stein's inequality), and both are isomorphic to their squares.

  $\implies$ by Pełczyński's decomposition method, $Y \simeq R_\beta^Y$.

Proof

- First, suppose that for some α, R^Y_α contains a complemented subspace isomorphic to Y.
- The *smallest* such α is a limit ordinal, so $Y \overset{c}{\hookrightarrow} (\bigoplus_{\beta<\alpha} R^Y_\beta)_{Y,\mathrm{ind}}$, but $Y \overset{c}{\not\hookrightarrow} R^Y_\beta$ for all $\beta < \alpha$.
- Ghawadrah 2016 $\implies$ there is an **independent** sequence in R^Y_α equivalent to the Haar system with closed span complemented in Y.
- Then such a system exists in any R^Y_β, $\omega \leq \beta < \omega_1$.
- Hence, $Y \overset{c}{\hookrightarrow} R^Y_\beta$, $R^Y_\beta \overset{c}{\hookrightarrow} Y$ (by Stein's inequality), and both are isomorphic to their squares.

  $\implies$ by Pełczyński's decomposition method, $Y \simeq R^Y_\beta$.

Proof

- First, suppose that for some α, R^Y_α contains a complemented subspace isomorphic to Y.
- The *smallest* such α is a limit ordinal, so $Y \overset{c}{\hookrightarrow} (\oplus_{\beta<\alpha} R^Y_\beta)_{Y,\mathrm{ind}}$, but $Y \overset{c}{\not\hookrightarrow} R^Y_\beta$ for all $\beta < \alpha$.
- Ghawadrah 2016 $\implies$ there is an **independent** sequence in R^Y_α equivalent to the Haar system with closed span complemented in Y.
- Then such a system exists in any R^Y_β, $\omega \leq \beta < \omega_1$.
- Hence, $Y \overset{c}{\hookrightarrow} R^Y_\beta$, $R^Y_\beta \overset{c}{\hookrightarrow} Y$ (by Stein's inequality), and both are isomorphic to their squares.

  $\implies$ by Pełczyński's decomposition method, $Y \simeq R^Y_\beta$.

Proof

- First, suppose that for some α, R^Y_α contains a complemented subspace isomorphic to Y.
- The *smallest* such α is a limit ordinal, so $Y \overset{c}{\hookrightarrow} (\oplus_{\beta<\alpha} R^Y_\beta)_{Y,\mathrm{ind}}$, but $Y \overset{c}{\not\hookrightarrow} R^Y_\beta$ for all $\beta < \alpha$.
- Ghawadrah 2016 $\implies$ there is an **independent** sequence in R^Y_α equivalent to the Haar system with closed span complemented in Y.
- Then such a system exists in any R^Y_β, $\omega \leq \beta < \omega_1$.
- Hence, $Y \overset{c}{\hookrightarrow} R^Y_\beta$, $R^Y_\beta \overset{c}{\hookrightarrow} Y$ (by Stein's inequality), and both are isomorphic to their squares.

  $\implies$ by Pełczyński's decomposition method, $Y \simeq R^Y_\beta$.

Proof

- First, suppose that for some α, R_α^Y contains a complemented subspace isomorphic to Y.
- The *smallest* such α is a limit ordinal, so $Y \stackrel{c}{\hookrightarrow} (\oplus_{\beta<\alpha} R_\beta^Y)_{Y,\mathrm{ind}}$, but $Y \stackrel{c}{\not\hookrightarrow} R_\beta^Y$ for all $\beta < \alpha$.
- Ghawadrah 2016 $\implies$ there is an **independent** sequence in R_α^Y equivalent to the Haar system with closed span complemented in Y.
- Then such a system exists in any R_β^Y, $\omega \leq \beta < \omega_1$.
- Hence, $Y \stackrel{c}{\hookrightarrow} R_\beta^Y$, $R_\beta^Y \stackrel{c}{\hookrightarrow} Y$ (by Stein's inequality), and both are isomorphic to their squares.

$\implies$ by Pełczyński's decomposition method, $Y \simeq R_\beta^Y$.

Proof

- First, suppose that for some α, R_α^Y contains a complemented subspace isomorphic to Y.
- The *smallest* such α is a limit ordinal, so $Y \overset{c}{\hookrightarrow} (\oplus_{\beta<\alpha} R_\beta^Y)_{Y,\mathrm{ind}}$, but $Y \overset{c}{\not\hookrightarrow} R_\beta^Y$ for all $\beta < \alpha$.
- Ghawadrah 2016 $\implies$ there is an **independent** sequence in R_α^Y equivalent to the Haar system with closed span complemented in Y.
- Then such a system exists in any R_β^Y, $\omega \leq \beta < \omega_1$.
- Hence, $Y \overset{c}{\hookrightarrow} R_\beta^Y$, $R_\beta^Y \overset{c}{\hookrightarrow} Y$ (by Stein's inequality), and both are isomorphic to their squares.

  $\implies$ by Pełczyński's decomposition method, $Y \simeq R_\beta^Y$.

Proof

- Now suppose that no space R_α^Y contains a complemented copy of Y.
- For every separable Banach space W, we define an ordinal index $i_Y[W] \in [0, \omega_1]$ such that:
  - $V \overset{c}{\hookrightarrow} W \implies i_Y[V] \le i_Y[W]$
  - $Y \overset{c}{\hookrightarrow} W \iff i_Y[W] = \omega_1$.
- Thus, $i_Y[R_\alpha^Y] < \omega_1$ for all α.
- Now, for $\xi < \omega_1$, inductively choose α_ξ sufficiently large such that $i_Y[R_{\alpha_\xi}^Y] > \sup_{\eta<\xi} i_Y[R_{\alpha_\eta}^Y]$.
- The spaces $R_{\alpha_\xi}^Y$ are pairwise non-isomorphic since they have pairwise distinct Y-indices.

Proof

- Now suppose that no space R_α^Y contains a complemented copy of Y.
- For every separable Banach space W, we define an ordinal index $i_Y[W] \in [0, \omega_1]$ such that:
  - $V \overset{c}{\hookrightarrow} W \implies i_Y[V] \leq i_Y[W]$
  - $Y \overset{c}{\hookrightarrow} W \iff i_Y[W] = \omega_1$.
- Thus, $i_Y[R_\alpha^Y] < \omega_1$ for all α.
- Now, for $\xi < \omega_1$, inductively choose α_ξ sufficiently large such that $i_Y[R_{\alpha_\xi}^Y] > \sup_{\eta<\xi} i_Y[R_{\alpha_\eta}^Y]$.
- The spaces $R_{\alpha_\xi}^Y$ are pairwise non-isomorphic since they have pairwise distinct Y-indices.

Proof

- Now suppose that no space R^Y_α contains a complemented copy of Y.
- For every separable Banach space W, we define an ordinal index $i_Y[W] \in [0, \omega_1]$ such that:
  - $V \overset{c}{\hookrightarrow} W \implies i_Y[V] \leq i_Y[W]$
  - $Y \overset{c}{\hookrightarrow} W \iff i_Y[W] = \omega_1$.
- Thus, $i_Y[R^Y_\alpha] < \omega_1$ for all α.
- Now, for $\xi < \omega_1$, inductively choose α_ξ sufficiently large such that $i_Y[R^Y_{\alpha_\xi}] > \sup_{\eta<\xi} i_Y[R^Y_{\alpha_\eta}]$.
- The spaces $R^Y_{\alpha_\xi}$ are pairwise non-isomorphic since they have pairwise distinct Y-indices.

Proof

- Now suppose that no space R^Y_α contains a complemented copy of Y.
- For every separable Banach space W, we define an ordinal index $i_Y[W] \in [0, \omega_1]$ such that:
    - $V \overset{c}{\hookrightarrow} W \implies i_Y[V] \leq i_Y[W]$
    - $Y \overset{c}{\hookrightarrow} W \iff i_Y[W] = \omega_1$.
- Thus, $i_Y[R^Y_\alpha] < \omega_1$ for all α.
- Now, for $\xi < \omega_1$, inductively choose α_ξ sufficiently large such that $i_Y[R^Y_{\alpha_\xi}] > \sup_{\eta<\xi} i_Y[R^Y_{\alpha_\eta}]$.
- The spaces $R^Y_{\alpha_\xi}$ are pairwise non-isomorphic since they have pairwise distinct Y-indices.

Proof

- Now suppose that no space R^Y_α contains a complemented copy of Y.
- For every separable Banach space W, we define an ordinal index $i_Y[W] \in [0, \omega_1]$ such that:
  - $V \overset{c}{\hookrightarrow} W \implies i_Y[V] \leq i_Y[W]$
  - $Y \overset{c}{\hookrightarrow} W \iff i_Y[W] = \omega_1$.
- Thus, $i_Y[R^Y_\alpha] < \omega_1$ for all α.
- Now, for $\xi < \omega_1$, inductively choose α_ξ sufficiently large such that $i_Y[R^Y_{\alpha_\xi}] > \sup_{\eta<\xi} i_Y[R^Y_{\alpha_\eta}]$.
- The spaces $R^Y_{\alpha_\xi}$ are pairwise non-isomorphic since they have pairwise distinct Y-indices.

Proof

- Now suppose that no space R^Y_α contains a complemented copy of Y.
- For every separable Banach space W, we define an ordinal index $i_Y[W] \in [0, \omega_1]$ such that:
  - $V \overset{c}{\hookrightarrow} W \implies i_Y[V] \leq i_Y[W]$
  - $Y \overset{c}{\hookrightarrow} W \iff i_Y[W] = \omega_1$.
- Thus, $i_Y[R^Y_\alpha] < \omega_1$ for all α.
- Now, for $\xi < \omega_1$, inductively choose α_ξ sufficiently large such that $i_Y[R^Y_{\alpha_\xi}] > \sup_{\eta<\xi} i_Y[R^Y_{\alpha_\eta}]$.
- The spaces $R^Y_{\alpha_\xi}$ are pairwise non-isomorphic since they have pairwise distinct Y-indices.

Proof

- Now suppose that no space R_α^Y contains a complemented copy of Y.
- For every separable Banach space W, we define an ordinal index $i_Y[W] \in [0, \omega_1]$ such that:
  - $V \overset{c}{\hookrightarrow} W \implies i_Y[V] \leq i_Y[W]$
  - $Y \overset{c}{\hookrightarrow} W \iff i_Y[W] = \omega_1$.
- Thus, $i_Y[R_\alpha^Y] < \omega_1$ for all α.
- Now, for $\xi < \omega_1$, inductively choose α_ξ sufficiently large such that $i_Y[R_{\alpha_\xi}^Y] > \sup_{\eta<\xi} i_Y[R_{\alpha_\eta}^Y]$.
- The spaces $R_{\alpha_\xi}^Y$ are pairwise non-isomorphic since they have pairwise distinct Y-indices.

Definition of complemented Y-index

For a separable Banach space W and $C \geq 1$, let $\mathscr{T}(Y, W, C)$ be the set of all finite sequences $((T_0, P_0), \ldots, (T_n, P_n))$ satisfying:

- T_k is an embedding from $\operatorname{span}\{h_I\}_{|I| \geq 2^{-k}}$ into W; $\|T_k\|, \|T_k^{-1}\| \leq C$.
- P_k is a projection from W onto the image of T_k with norm $\leq C$.
- Some compatibility conditions.

$\mathscr{T}(Y, W, C)$ is a tree. The index $i_Y[W]$ is essentially the *order* of this tree as $C \to \infty$:

$$i_Y[W] := \begin{cases} \omega_1 & \text{if } \mathscr{T}(Y, W, C) \text{ ill-founded for some } C. \\ \sup_{C \geq 1} o[\mathscr{T}(Y, W, C)] & \text{otherwise.} \end{cases}$$

Definition of complemented Y-index

For a separable Banach space W and $C \geq 1$, let $\mathscr{T}(Y, W, C)$ be the set of all finite sequences $((T_0, P_0), \dots, (T_n, P_n))$ satisfying:

- T_k is an embedding from $\operatorname{span}\{h_I\}_{|I| \geq 2^{-k}}$ into W; $\|T_k\|, \|T_k^{-1}\| \leq C$.
- P_k is a projection from W onto the image of T_k with norm $\leq C$.
- Some compatibility conditions.

$\mathscr{T}(Y, W, C)$ is a tree. The index $i_Y[W]$ is essentially the *order* of this tree as $C \to \infty$:

$$i_Y[W] := \begin{cases} \omega_1 & \text{if } \mathscr{T}(Y, W, C) \text{ ill-founded for some } C. \\ \sup\limits_{C \geq 1} o[\mathscr{T}(Y, W, C)] & \text{otherwise.} \end{cases}$$

Definition of complemented Y-index

For a separable Banach space W and $C \geq 1$, let $\mathscr{T}(Y, W, C)$ be the set of all finite sequences $((T_0, P_0), \ldots, (T_n, P_n))$ satisfying:

- T_k is an embedding from $\operatorname{span}\{h_I\}_{|I| \geq 2^{-k}}$ into W; $\|T_k\|, \|T_k^{-1}\| \leq C$.
- P_k is a projection from W onto the image of T_k with norm $\leq C$.
- Some compatibility conditions.

$\mathscr{T}(Y, W, C)$ is a tree. The index $i_Y[W]$ is essentially the *order* of this tree as $C \to \infty$:

$$i_Y[W] := \begin{cases} \omega_1 & \text{if } \mathscr{T}(Y, W, C) \text{ ill-founded for some } C. \\ \sup\limits_{C \geq 1} o[\mathscr{T}(Y, W, C)] & \text{otherwise.} \end{cases}$$

Definition of complemented Y-index

For a separable Banach space W and $C \geq 1$, let $\mathscr{T}(Y, W, C)$ be the set of all finite sequences $((T_0, P_0), \ldots, (T_n, P_n))$ satisfying:

- T_k is an embedding from $\operatorname{span}\{h_I\}_{|I| \geq 2^{-k}}$ into W; $\|T_k\|, \|T_k^{-1}\| \leq C$.
- P_k is a projection from W onto the image of T_k with norm $\leq C$.
- Some compatibility conditions.

$\mathscr{T}(Y, W, C)$ is a tree. The index $i_Y[W]$ is essentially the *order* of this tree as $C \to \infty$:

$$i_Y[W] := \begin{cases} \omega_1 & \text{if } \mathscr{T}(Y, W, C) \text{ ill-founded for some } C. \\ \sup_{C \geq 1} o[\mathscr{T}(Y, W, C)] & \text{otherwise.} \end{cases}$$

Definition of complemented Y-index

For a separable Banach space W and $C \geq 1$, let $\mathscr{T}(Y, W, C)$ be the set of all finite sequences $((T_0, P_0), \ldots, (T_n, P_n))$ satisfying:

- T_k is an embedding from $\mathrm{span}\{h_I\}_{|I| \geq 2^{-k}}$ into W; $\|T_k\|, \|T_k^{-1}\| \leq C$.
- P_k is a projection from W onto the image of T_k with norm $\leq C$.
- Some compatibility conditions.

$\mathscr{T}(Y, W, C)$ is a tree. The index $i_Y[W]$ is essentially the *order* of this tree as $C \to \infty$:

$$i_Y[W] := \begin{cases} \omega_1 & \text{if } \mathscr{T}(Y, W, C) \text{ ill-founded for some } C. \\ \sup\limits_{C \geq 1} o[\mathscr{T}(Y, W, C)] & \text{otherwise.} \end{cases}$$

Definition of complemented Y-index

For a separable Banach space W and $C \geq 1$, let $\mathscr{T}(Y, W, C)$ be the set of all finite sequences $((T_0, P_0), \ldots, (T_n, P_n))$ satisfying:

- T_k is an embedding from $\mathrm{span}\{h_I\}_{|I| \geq 2^{-k}}$ into W; $\|T_k\|, \|T_k^{-1}\| \leq C$.
- P_k is a projection from W onto the image of T_k with norm $\leq C$.
- Some compatibility conditions.

$\mathscr{T}(Y, W, C)$ is a tree. The index $i_Y[W]$ is essentially the *order* of this tree as $C \to \infty$:

$$i_Y[W] := \begin{cases} \omega_1 & \text{if } \mathscr{T}(Y, W, C) \text{ ill-founded for some } C. \\ \sup_{C \geq 1} o[\mathscr{T}(Y, W, C)] & \text{otherwise.} \end{cases}$$

Definition of complemented Y-index

For a separable Banach space W and $C \geq 1$, let $\mathscr{T}(Y, W, C)$ be the set of all finite sequences $((T_0, P_0), \dots, (T_n, P_n))$ satisfying:

- T_k is an embedding from $\operatorname{span}\{h_I\}_{|I| \geq 2^{-k}}$ into W; $\|T_k\|, \|T_k^{-1}\| \leq C$.
- P_k is a projection from W onto the image of T_k with norm $\leq C$.
- Some compatibility conditions.

$\mathscr{T}(Y, W, C)$ is a tree. The index $i_Y[W]$ is essentially the *order* of this tree as $C \to \infty$:

$$i_Y[W] := \begin{cases} \omega_1 & \text{if } \mathscr{T}(Y, W, C) \text{ ill-founded for some } C. \\ \sup\limits_{C \geq 1} o[\mathscr{T}(Y, W, C)] & \text{otherwise.} \end{cases}$$

Further results

Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a strictly increasing Young's function with $\Phi(s) \sim s(\log s)^\alpha$ as $s \to \infty$, for some $0 < \alpha < 1/2$.

Theorem (Konstantos, Motakis, S. '26)

Let $Y = L^1_0$, H^1, the Orlicz space L^Φ_0, or H_{L^Φ}.

- *The space R^Y_α is uncomplemented in Y for $\omega \cdot 2 \le \alpha < \omega_1$.*
- *For every $n \in \mathbb{N}_0$, there exist realizations of $R^Y_{\omega+n}$ that are uncomplemented in Y.*
- *But there exist realizations of $R^{H^1}_\omega$ that are complemented in H^1 (Müller-Schechtman 1989).*

Further results

Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a strictly increasing Young's function with $\Phi(s) \sim s(\log s)^\alpha$ as $s \to \infty$, for some $0 < \alpha < 1/2$.

Theorem (Konstantos, Motakis, S. '26)

Let $Y = L^1_0$, H^1, the Orlicz space L^Φ_0, or H_{L^Φ}.

- *The space R^Y_α is uncomplemented in Y for $\omega \cdot 2 \leq \alpha < \omega_1$.*
- *For every $n \in \mathbb{N}_0$, there exist realizations of $R^Y_{\omega+n}$ that are uncomplemented in Y.*
- *But there exist realizations of $R^{H^1}_\omega$ that are complemented in H^1 (Müller-Schechtman 1989).*

Further results

Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a strictly increasing Young's function with $\Phi(s) \sim s(\log s)^\alpha$ as $s \to \infty$, for some $0 < \alpha < 1/2$.

Theorem (Konstantos, Motakis, S. '26)

Let $Y = L^1_0$, H^1, the Orlicz space L^Φ_0, or H_{L^Φ}.

- *The space R^Y_α is uncomplemented in Y for $\omega \cdot 2 \le \alpha < \omega_1$.*
- *For every $n \in \mathbb{N}_0$, there exist realizations of $R^Y_{\omega+n}$ that are uncomplemented in Y.*
- *But there exist realizations of $R^{H^1}_\omega$ that are complemented in H^1 (Müller-Schechtman 1989).*

Further results

Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a strictly increasing Young's function with $\Phi(s) \sim s(\log s)^\alpha$ as $s \to \infty$, for some $0 < \alpha < 1/2$.

Theorem (Konstantos, Motakis, S. '26)

Let $Y = L^1_0$, H^1, the Orlicz space L^Φ_0, or H_{L^Φ}.

- *The space R^Y_α is uncomplemented in Y for $\omega \cdot 2 \leq \alpha < \omega_1$.*
- *For every $n \in \mathbb{N}_0$, there exist realizations of $R^Y_{\omega+n}$ that are uncomplemented in Y.*
- *But there exist realizations of $R^{H^1}_\omega$ that are complemented in H^1 (Müller-Schechtman 1989).*

Further results

Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a strictly increasing Young's function with $\Phi(s) \sim s(\log s)^\alpha$ as $s \to \infty$, for some $0 < \alpha < 1/2$.

Theorem (Konstantos, Motakis, S. '26)

Let $Y = L^1_0$, H^1, the Orlicz space L^Φ_0, or H_{L^Φ}.

- *The space R^Y_α is uncomplemented in Y for $\omega \cdot 2 \leq \alpha < \omega_1$.*
- *For every $n \in \mathbb{N}_0$, there exist realizations of $R^Y_{\omega+n}$ that are uncomplemented in Y.*
- *But there exist realizations of $R^{H^1}_\omega$ that are complemented in H^1 (Müller-Schechtman 1989).*

Thank you for your attention!