Weakly almost square Lipschitz-free spaces

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Lipschitz spaces and Lipschitz-free spaces

Let M be a metric space with a fixed point 0. The set of all Lipschitz functions f with $f(0) = 0$ is a Banach space with the norm

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An element in $\mathcal{F}(M)$ of the form

$$m_{xy} = \frac{\delta_x - \delta_y}{d(x,y)}$$

for $x \neq y \in M$ is called a **molecule**. It is well-known that $\|m_{xy}\| = 1$.

Almost square properties

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$$\mathrm{ASQ} \underset{\not\Leftarrow}{\Rightarrow} \mathrm{WASQ} \underset{\not\Leftarrow}{\Rightarrow} \mathrm{LASQ}$$

$$L_1[0,1] \qquad\qquad \mathcal{F}(M)$$

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Let $M := [0,1]^2$ be a metric space with the metric

$$d\big((a,b),(c,d)\big) := \begin{cases} |a-c|, & \text{if } b = d, \\ \min\{a+c, 2-a-c\} + |b-d|, & \text{if } b \neq d. \end{cases}$$

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The element $\mu \in S_{\mathcal{F}(M)}$, defined by

$$\mu(f) = \int_0^1 \big(f(1,t) - f(0,t)\big)\,dt, \quad f \in Lip_0(M).$$

is not a WASQ point.

Open questions

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