Subideals of $\mathcal{L}(X)$ of higher order

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Aim of talk

- Let X be a real or complex Banach space, and $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators $X \to X$.

Aim of talk: To introduce the notion of a non-trivial n-subideal of $\mathcal{L}(X)$ and to present two examples with decreasing sequences of such n-subideals.

The talk is based on joint work with Hans-Olav Tylli (Helsinki):

- H-O. Tylli, H. Wirzenius: *Closed subideals of $L(X)$ of higher order*, preprint.
- H-O. Tylli, H. Wirzenius: *Structure of closed subideals of $L(X)$*, preprint.

Starting point

- Recall: a closed linear subspace $\mathcal{J}$ of a Banach algebra $\mathcal{A}$ is a closed ideal of $\mathcal{A}$ if $ST \in \mathcal{J}$ and $TS \in \mathcal{J}$ for all $S \in \mathcal{J}$ and $T \in \mathcal{A}$.
- For Banach space X we will consider the following closed ideals of $\mathcal{L}(X)$:

$$\mathcal{K}(X) = \{\text{compact operators } X \to X\}$$
$$\mathcal{S}(X) = \{\text{strictly singular operators } X \to X\}$$

  - ⋆ T is strictly singular if the restriction of T does not define a linear isomorphism $M \to TM$ for any closed ∞-dim. subspace $M \subset X$.
- A result of Chernoff (JFA 1973) implies that two distinct closed ideals of $\mathcal{L}(X)$ are never isomorphic as Banach algebras.
- In (Tylli & W., Proc. AMS 2024) we showed that there is a Banach space X failing the AP that carries an uncountable family of distinct closed ideals of $\mathcal{K}(X)$ which are pairwise isomorphic as Banach algebras.
  - ⋆ Similar result was obtained for closed ideals of $\mathcal{S}(X)$ for classical spaces including $C(0,1)$ and $L_p(0,1)$, $p \neq 2$.

  Hence Chernoff's result fails in general for closed subideals of $\mathcal{L}(X)$ (meaning closed ideals of closed ideals of $\mathcal{L}(X)$).
- What can be said about subideals of $\mathcal{L}(X)$ of higher order?

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$$\mathcal{J} \rhd \mathcal{A}.$$

Let $\mathcal{J} \subset \mathcal{L}(X)$ be a subspace and $n \in \mathbb{N}$. We call $\mathcal{J}$ an **algebraic** ***n*-subideal of** $\mathcal{L}(X)$ if there are subalgebras $\mathcal{J}_0, \ldots, \mathcal{J}_n \subset \mathcal{L}(X)$ such that

$$\mathcal{J} = \mathcal{J}_n \rhd \mathcal{J}_{n-1} \rhd \cdots \rhd \mathcal{J}_1 \rhd \mathcal{J}_0 = \mathcal{L}(X) \tag{1}$$

(that is, $\mathcal{J}_k$ is an ideal of $\mathcal{J}_{k-1}$ for all $k = 1, \ldots, n$).
Above $\mathcal{J}$ is called a **closed** ***n*-subideal of** $\mathcal{L}(X)$ if the subalgebras $\mathcal{J}$ and $\mathcal{J}_k$ in (1) are closed.

- ideal = 1-subideal.
- If $\mathcal{J}$ is an n-subideal of $\mathcal{L}(X)$, then $\mathcal{J}$ is an $(n+1)$-subideal of $\mathcal{L}(X)$. (Reason: Add $\mathcal{J}_{n+1} := \mathcal{J}_n$ in (1).)

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- We are interested in the following natural question:
  - Q. *For a given Banach space X and $n \geq 2$, are there closed n-subideals of $\mathcal{L}(X)$ that are not closed $(n-1)$-subideals of $\mathcal{L}(X)$?*
  - ⋆ We call such closed n-subideals **non-trivial**.
  - ⋆ Similarly, we call an algebraic n-subideal of $\mathcal{L}(X)$ non-trivial if it is not an algebraic $(n-1)$-subideal of $\mathcal{L}(X)$.
- Non-trivial closed n-subideals of $\mathcal{L}(X)$ do not always exist:
  For example,
  - ⋆ if H is a Hilbert space, then $\mathcal{L}(H)$ does not have any non-trivial closed n-subideals.

    Reason: every closed ideal of $\mathcal{L}(H)$ possesses an a.i.
  - ⋆ if $X = c_0$ or $X = \ell_p$, $1 \leq p < \infty$.

    Reason:
    - $\mathcal{K}(X)$ is the only proper non-zero closed ideal of $\mathcal{L}(X)$.
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  We are not aware of any systematic studies of n-subideals of Banach algebras $\mathcal{L}(X)$ (or more generally for Banach algebras A) for $n \geq 3$.
- For $n = 2$, Fong and Radjavi (Math. Ann., 1983) obtained non-trivial algebraic 2-subideals of $\mathcal{L}(\ell_2)$:
  - ⋆ For example, let $[D_s]_{\mathcal{K}}$ denote the ideal of $\mathcal{K}(\ell_2)$ generated by the diagonal operator $D_s \in \mathcal{K}(\ell_2)$ where $s = (1/n^{\alpha})_{n=1}^{\infty}$ for fixed $\alpha > 0$. Then $[D_s]_{\mathcal{K}}$ is not an ideal of $\mathcal{L}(\ell_2)$.
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  - ⋆ For example, let $[D_s]_{\mathcal{K}}$ denote the ideal of $\mathcal{K}(\ell_2)$ generated by the diagonal operator $D_s \in \mathcal{K}(\ell_2)$ where $s = (1/n^\alpha)_{n=1}^\infty$ for fixed $\alpha > 0$. Then $[D_s]_{\mathcal{K}}$ is not an ideal of $\mathcal{L}(\ell_2)$.
- In the theory of Lie algebras there is a parallel concept of Lie n-subideals, which goes back to (Schenkman, Amer. J. Math., 1951) (as subinvariant subalgebras).

Basic observations

- Let $\mathcal{I}$ be a closed ideal of $\mathcal{L}(X)$ and $\mathcal{A} \subset \mathcal{I}$ a subalgebra. Put $\mathcal{M}_n = \overline{\mathcal{A} + \mathcal{I}^n}$ for $n \geq 2$. Then

$$\mathcal{M}_n \triangleright \mathcal{M}_{n-1} \triangleright \cdots \triangleright \mathcal{M}_2 \triangleright \mathcal{I} \triangleright \mathcal{L}(X).$$

  so $\mathcal{M}_n$ is a closed n-subideal of $\mathcal{L}(X)$.
- Recall: If $\mathcal{M}_n$ is a closed n-subideal of $\mathcal{L}(X)$, then $\overline{\mathcal{F}(X)} \subset \mathcal{M}_n$.
- Hence, if $\mathcal{I}^n \subset \overline{\mathcal{F}(X)}$, then every closed subalgebra

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  is a closed n-subideal of $\mathcal{L}(X)$.
- **Consequence:** If $\mathcal{M}_n$ is a non-trivial closed n-subideals of $\mathcal{L}(X)$, the ideal $\mathcal{I} \subset \mathcal{L}(X)$ in the chain

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Infinite chain of non-trivial closed n-subideals

Theorem A. Let $X = \left(\ell_{p_1} \oplus \ell_{p_2} \oplus \cdots\right)_{c_0}$, where $1 \leq p_1 < p_2 < \ldots < \infty$. Then there is a decreasing chain

$$(\mathcal{K}(X) \subset) \quad \cdots \rhd \mathcal{M}_n \rhd \cdots \rhd \mathcal{M}_2 \rhd \mathcal{S}(X) \rhd \mathcal{L}(X)$$

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Here $\mathcal{M}_n = \overline{\mathcal{A} + \mathcal{S}(X)^n}$ for all $n \in \mathbb{N}$, where $\mathcal{A} \subset \mathcal{S}(X)$ is the subalgebra spanned by operators $V_1, V_2, \ldots$ defined as follows:

$$\begin{tikzcd}
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\end{tikzcd}$$

where $S_k \in \mathcal{S}(X_k)$ such that $S_k^{k-1} \notin \mathcal{K}(X_k)$.

Key observation for non-triviality: $P_n U J_n \in \mathcal{K}(X_n)$ for all $U \in \mathcal{S}(X)^n$.

Application of the notion "matrix multiplication works" (Laustsen, Math. Proc. Camb. Phil. Soc. 2001).

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Algebraic n-subideals

- Recall: Fong & Radjavi obtained non-trivial algebraic 2-subideals of $\mathcal{L}(\ell_2)$ inside the compact operators $\mathcal{K}(\ell_2)$.
- There are even chains

$$\cdots \triangleright \mathcal{J}_n \triangleright \cdots \triangleright \mathcal{J}_1 \triangleright \mathcal{L}(\ell_2)$$

where each algebraic n-subideal $\mathcal{J}_n$ is non-trivial!

  - $\star$ Recall: there are no non-trivial closed n-subideals of $\mathcal{L}(\ell_2)$.

- For this, we require the class of entropy ideals for $1 < p < \infty$:

$$\mathcal{L}_p^{(e)}(X) := \{S \in \mathcal{L}(X) \mid (e_n(S)) \in \ell_p\} \triangleright \mathcal{L}(X),$$

where

$$e_n(S) = \inf\{\varepsilon > 0 \mid SB_X \subset \{y_1, \ldots, y_q\} + \varepsilon B_X, q \leq 2^{n-1}\}$$

is the n:th entropy number of an operator $S \in \mathcal{L}(X)$.

- Key fact (Gordon, König, Schütt, J. Approx. Theory, 1987): Suppose X has an unconditional basis (e_k) and let $t_1 \geq t_2 \geq \cdots > 0$. For the diagonal operator

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Theorem B. Let $1 < p < \infty$ and let X be a Banach space with a normalized unconditional Schauder basis (e_k). Put $Z = X \oplus X$. Then there is a decreasing chain

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where $\mathcal{K}_n$ is a non-trivial algebraic n-subideal of $\mathcal{L}(Z)$ for $n \geq 2$.

Construction of $\mathcal{K}_n$: Pick sequence

$$t_p \in \ell_p \setminus \bigcup_{q<p} \ell_q.$$

Let $D_{t_p} : X \to X$ be the diagonal operator $e_k \mapsto t_p(k)e_k$. Define

$$U_p = \begin{bmatrix} D_{t_p} & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{L}_p^{(e)}(Z)$$

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$$\mathcal{K}_n = span\{U_p^k \mid 1 \leq k < n\} + \mathcal{L}_p^{(e)}(Z)^n, \quad n \geq 2.$$

Algebraic n-subideals, II

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- Recall Theorem A: for $X = (\ell_{p_1} \oplus \ell_{p_2} \oplus \cdots)_{c_0}$ there is a chain

$$\cdots \rhd \mathcal{M}_n \rhd \mathcal{M}_{n-1} \rhd \cdots \rhd \mathcal{M}_1 \rhd \mathcal{L}(X)$$

where $\mathcal{M}_n$ is a non-trivial closed n-subideal of $\mathcal{L}(X)$ for all $n \geq 2$.

Q1. **Are there other classes of Banach spaces X with such chains?**

- Remark: For fixed $n \geq 2$ it is somewhat easier to find Banach spaces X that carry non-trivial closed n-subideals of $\mathcal{L}(X)$; that is, with a finite chain

$$\mathcal{M}_n \rhd \mathcal{M}_{n-1} \rhd \cdots \rhd \mathcal{M}_1 \rhd \mathcal{L}(X)$$

where $\mathcal{M}_n$ is a non-trivial closed n-subideal.

Sample examples: For $1 < p_1 < p_2 < \cdots < p_n < \infty$, the finite direct sums

- ⋆ $X = \bigoplus_{j=1}^{n} \ell_{p_j}$
- ⋆ $X \oplus X$ for $X = \bigoplus_{j=1}^{n} J_{p_j}$ (p-James spaces)
- ⋆ $X = \bigoplus_{j=1}^{n-1} S_{p_j}$ (p-convexified Schreier spaces)

Q2. **Are there chains of non-trivial closed n-subideals for other classes of Banach algebras?**

- Thank you for your attention.

- Recall Theorem A: for $X = (\ell_{p_1} \oplus \ell_{p_2} \oplus \cdots)_{c_0}$ there is a chain

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where $\mathcal{M}_n$ is a non-trivial closed n-subideal of $\mathcal{L}(X)$ for all $n \geq 2$.

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Sample examples: For $1 < p_1 < p_2 < \cdots < p_n < \infty$, the finite direct sums

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- $\star$ $X \oplus X$ for $X = \bigoplus_{j=1}^{n} J_{p_j}$ (p-James spaces)
- $\star$ $X = \bigoplus_{j=1}^{n-1} S_{p_j}$ (p-convexified Schreier spaces)

Q2. **Are there chains of non-trivial closed n-subideals for other classes of Banach algebras?**

- Thank you for your attention.

Concluding questions

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